Q.22. How to configure Exponential Smoothing? Discuss

Answer:

1. Need to specify all the model hyperparameters explicitly. This can be challenging for both beginners and experts.

2. Numerical optimization is commonly used to search for and find the smoothing factors (α, β and γ) for the model resulting in the most negligible error.

3. An exponential smoothing method can obtain values for unknown parameters by estimating them from the observed data.

4. The initial values and unknown parameters can be estimated by minimizing sum of squared errors.

3rd Tutorial

Q1. What is a linear process? Give some examples of such process

Answer:

The time series $\{X_t\}$ is a linear process if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

for all t, where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constant with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

Examples:

1) Autoregressive (AR) process.
2) Moving average (MA) process.
3) Autoregressive Moving Average (ARMA) process.
4) Autoregressive Integrated Moving Average (ARIMA) process.

Q2. Define an ARMA (P, q) process when it is called i) invertible and (ii) non-invertible process?

Answer:

$\{X_t\}$ is an ARMA (P; q) process if $\{X_t\}$ is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials ~~$(1-\theta_1 z +$~~ $(1-\phi_1 z - \cdots - \phi_p z^p)$ and $(1+\theta_1 z + \cdots + \theta_q z^q)$ have no common factors.

i) An ARMA (p,q) process $\{X_t\}$ is invertible if there exist constant $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all t.

Invertibility is equivalent to the condition

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \quad \text{for all } |z| \leq 1$$

ii) An ARMA (p,q) process is non-invertible when the roots of the MA polynomial lie inside the unit circle.

Q3. When an ARMA process in considered as a causal process?

Answer:

An ARMA (p, q) process $\{X_t\}$ is causal, or a causal function of $\{Z_t\}$, if there exist constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \text{for all } t$$

Causality is equivalent to the condition-

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \quad \text{for all } |z| \leq 1$$

~~Q4. When an ARMA (1,1) process is considered as an cattle~~

Q4. When an ARMA (1, 1) process in considered as (i) invertible and (ii) non invertible process?

Answer:

i) If $|\theta| < 1$, then the ARMA (1,1) process is invertible and Z_t is expressed in terms of X_s, $s \leq t$; where

$$Z_t = X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}.$$

ii) If $|\theta| > 1$ then the ARMA(1,1) process is noninvertible and Z_t is expressed in terms of $X_s,\ s \geq t$ where

$$Z_t = -\phi\theta^{-1} X_t + (\theta+\phi) \sum_{j=1}^{\infty} (-\theta)^{-j-1} X_{t+j}$$

Q5. Check whether the function is the ACVF of a stationary process.

Answer:

A function $\gamma(\cdot)$ defined on the integers is the ACVF of a stationary time series if and only if there exists a right-continuous, nondecreasing, bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$ such that

$$\gamma(h) = \int_{(-\pi,\pi)} e^{ih\lambda}\, dF(\lambda)$$

for all integers h.

Q6. When an ARMA process is considered as an invertible process?

Answer:

An ARMA (p, q) process $\{X_t\}$ is invertible if there exist constants $\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad \text{for all } t$$

Invertibility is equivalent to the condition

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \quad \text{for all } |z| \leq 1$$

Q7. Check whether or not the following process is unique, causal and invertible, where ~~$\{Z_t\} \sim WN(0,$~~

$\{Z_t\} \sim WN(0, \sigma^2)$

i) $X_t + 0.2 X_{t-1} - 0.48 X_{t-2} = Z_t$,

ii) $X_t + 0.6 X_{t-1} = Z_t + 1.2 Z_{t-1}$,

iii) $X_t + 1.8 X_{t-1} + 0.81 X_{t-2} = Z_t$

Solution (i):

(i) Here the autoregressive polynomial is

$\phi(z) = 1 + 0.2z - 0.48z^2$

$= (1+0.8z)(1-0.6z)$

the autoregressive polynomial is zero at

$z = -1/0.8 = -1.25$ or $z = 1/0.6 = 1.67$

Since both of these are located outside the unit circle, the process is causal & AR(2) process that is also unique and the coefficients $\{\psi_j\}$ given by

$\psi_0 = 1$

$\psi_1 = -0.2$

$\psi_2 = (-0.2)^2 + 0.48$

$\psi_j = (-0.2)\psi_{j-1} + 0.48\psi_{j-2}$; $j = 2, 3, \ldots$

Process is obviously invertible since ~~$Z_t = X_t + 0.2X$~~

$Z_t = X_t + 0.2X_{t-1} - 0.48X_{t-2}$.

(ii): Let $\{X_t\}$ be ARMA (1, 1) process,

$$X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1} \;;\; \{Z_t\} \sim WN(0, \sigma^2)$$

Since the autoregressive polynomial $\phi(z) = 1 + 0.6z$ has a zero at $z = -1/.6 = -1.67$ on $|z| > 1$ which is located outside the unit circle, we conclude that there exists a unique ARMA process that is also causal. the coefficients $\{\psi_j\}$ is given b

$\psi_0 = 1$

$\psi_1 = (1.2 - 0.6)$

$\psi_2 = -0.6(1.2 - 0.6)$

$\psi_j = (-0.6)^{j-1}(1.2 - 0.6) \quad ; \; j = 1, 2, \ldots$

The MA polynomials $\theta(z) = 1 + 1.2z$ has a zero at $z = -1/1.2 = -0.83$ which is located inside the unit circle, then $\{X_t\}$ is not invertible.

(iii) Let $\{X_t\}$ be the AR(1) process,

X_t ~~$+ 1.8X_{t-1} + 0.8X_{t-1} + 0.8$~~

$$X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t \;,\; \{Z\} \sim WN(0, \sigma^2)$$

The autoregressive polynomial $\phi(z) = 1 + 1.8z + 0.81z^2$ $= (1-0.9)(1-0.9)$ and is therefore zero at $z = -10/9$ $= -1.11$

hence these zero lies outside the unit circle we conclude that there exists a unique AR(2) process that is also causal. the coefficients $\{\psi_j\}$ is given by

$\psi_0 = 1$

$\psi_1 = -1.8$

$\psi_2 = (-1.8)^2 - 0.81$

$\psi_j = (-1.8)\psi_{j-1} - 0.81\psi_{j-2}$; $j = 2, 3, \dots$

and also invertible.

Q8. Find the mean and autocovariance function of the following process:

i) $X_t = Z_t - 0.7Z_{t-1} + 0.5Z_{t-3}$

ii) $X_t = Z_t Z_{t-2}$

iii) $X_t = Z_t + 0.8Z_{t-3}$

where $\{Z_t\} \sim WN(0, 1)$. Hence, check the stationarity of the process.

Solution (i):

$$X_t = Z_t - 0.7Z_{t-1} + 0.5Z_{t-3}$$

$$\Rightarrow E[X_t] = E[Z_t - 0.7Z_{t-1} + 0.5Z_{t-3}]$$

$$= E[Z_t] - 0.7E[Z_{t-1}] + 0.5E[Z_{t-3}]$$

~~Since~~ Since, $E[Z_t] = 0$

So, $E[Z_{t-1}] = E[Z_{t-3}] = 0$

$\therefore E(X_t) = 0$

$\gamma(t+h, t) = \gamma(h)$

$= cov(X_{t+h}, X_t)$

$= E[X_{t+h}, X_t]$ since $E(X_t) = E(X_{t+h}) = 0$

when, $h = 0$

$\gamma(0) = E[X_t, X_t]$

$= E[X_t^2]$

$= E\left[\left(Z_t - 0.7Z_{t-1} + 0.5Z_{t-3}\right)^2\right]$

$= E[Z_t^2] + (0.7)^2 E[Z_{t-1}^2] + (0.5)^2 E[Z_{t-3}^2]$

$= 1 + (0.7)^2 \times 1 + (0.5)^2 \times 1$

$= 1 + 0.74$

$\therefore \gamma(0) = 1.74$

$\gamma(h) = E[X_t, X_{t+h}]$

$$X_{t+h} = Z_{t+h} - 0.7Z_{t-1+h} + 0.5Z_{t-3+h}$$

$$= Z_{t+h} - 0.7Z_{t+h-1} - 0.5Z_{t+h-3}$$

$$\therefore \gamma(h) = E[Z_t Z_{t+h}] - 0.7[Z_t Z_{t+h-1}] + 0.5[Z_t Z_{t+h-3}] -$$

$$0.7 E[Z_{t-1} \cdot Z_{t+h-1}] + (0.7)^2 E[Z_{t-1} \cdot Z_{t+h-1}]$$

$$(0.5 \times 0.7) E[Z_{t-1} Z_{t+h-3}] + 0.5 E[Z_{t-3} Z_{t+h}] -$$

$$(0.7 \times 0.5) \cancel{E[Z_{t-3} Z_{t+h}]} E[Z_{t-3}, Z_{t+h-1}] + (0.5)^2$$

$$E[Z_{t-3} . Z_{t+h}]$$

$\gamma(0) = 1.74$

$\gamma(1) = 0 - 0.7 E[Z_t^2] = -0.7 \times 1 = -0.7$

So, the process is stationary.

(ii) $X_t = Z_t Z_{t-2}$

Mean $= E[X_t] = E[Z_t Z_{t-2}] = 0$

ACVF $\gamma(h) = \text{cov}(X_{t+h}, X_t)$

$$= E(X_{t+h}, X_t)$$

$$= E[(Z_{t+h} \cdot Z_{t+h-2})(Z_t Z_{t-2})]$$

$$- E[Z_t \cdot Z_{t-2}] \cdot E[Z_{t+h} \cdot Z_{t+h-2}]$$

$$\text{cov}(X_t, X_{t+h}) = E[Z_t \cdot Z_{t-2}]$$

$$= E[Z_{t+h} \cdot Z_{t+h-2}]$$

$$= 0 + 0$$

$$= 0$$

$$\cancel{\gamma(0) = E[Z_t^2 Z_{t-2}^2]}$$

$$\cancel{= E[Z_t^2 Z_{t-2}^2]}$$

$$\gamma(0) = E[Z_t^2 \cdot Z_{t-2}^2]$$

$$= E[Z_t^2] \cdot E[Z_{t-2}^2]$$

$$= 1 \cdot 1$$

$$\therefore \gamma(0) = 1$$

$$\therefore \gamma(h) = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

(iii) $X_t = Z_t + 0.8 Z_{t-3}$

Mean, $E(X_t) = E[Z_t + 0.8 Z_{t-3}]$

$$= E[Z_t] + 0.8 E[Z_{t-3}]$$

$$= 0 + 0.8 \times 0$$

$$= 0$$

(P.T.O)

$$\gamma(h) = cov(X_{t+h}, X_t)$$
$$= E[X_{t+h}, X_t]$$
$$= E[(Z_{t+h} + 0.8 Z_{t+h-3})(Z_t + 0.8 Z_{t-3})]$$
$$= E[Z_{t+h}(Z_t + 0.8 Z_{t-3}) + 0.8 (Z_{t+h-3})(Z_t + 0.8 Z_{t-3})]$$
$$= E[Z_{t+h} Z_t + 0.8 Z_{t+h} Z_{t-3} + 0.8 Z_t \cdot Z_{t+h-3} + (0.8)^2 Z_{t+h-3} Z_{t-3}]$$

$$\therefore \gamma(h) = E[Z_{t+h} \cdot Z_t] + 0.8 E[Z_{t+h} Z_{t-3}] + 0.8 E[Z_t \cdot Z_{t+h-3}] + (0.8)^2 E[Z_{t+h-3} Z_{t-3}]$$

if $h=0$, $\gamma(0) = E[Z_t^2] + (0.8)^2 E[Z_{(t-3)}^2]$
$$= 1 + (0.8)^2 \times 1$$
$$= \boxed{1.64}$$

$h = 1$, $\gamma(1) = 0$

$h=-3$, $\gamma(-3) = 0.8 E[Z_{t-3}^2]$
$$= 0.8 \times 1$$
$$= 0.8$$

Q9. What are the different methods to calculate ACVF of a time series? Calculate the ACVF of the MA(2) Process:

(i) $X_t = Z_t + 0.3Z_{t-1} - 1.2Z_{t-2}$

(ii) $X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}$

$\{Z_t\} \sim WN(0, 1)$

Solution:

The Methods of calculating ACVF are

1) Direct method.
2) Sample autocovariance function.
3) ACF Plot.
4) Yule-walker equation.
5) Fast Fourier transform (FFT).
6) Parametric modeling.
7) Cross covariance function.

i) Given,

$X_t = Z_t + 0.3Z_{t-1} - 1.2Z_{t-2}$

MA(2)

$$\gamma(h) = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma^2 & ; h=0 \\ (\theta_1+\theta_1\theta_2)\sigma^2 & ; h=\pm1 \\ \theta_2\sigma^2 & ; h=\pm2 \\ & ; |h|>1 \end{cases}$$

$$\gamma(0) = [1 + (0.3)^2 + (-1.2)^2]\sigma^2$$

$$= 2.53 \times 1 = 2.53$$

$$\gamma(1) = [0.3 + (0.3)(-1.2)] \times 1$$

$$= -0.06$$

$$\gamma(2) = (-1.2) \times 1 = -1.2$$

$$\gamma(h) = \begin{cases} 2.53 & \text{for } h=0 \\ -0.06 & \text{for } h=\pm 1 \\ -1.2 & \text{for } h=\pm 2 \\ 0 & \text{for } |h|>2 \end{cases}$$

$$\rho(h) = \begin{cases} 1 & \text{for } h=0 \\ -0.02 & \text{for } h=\pm 1 \\ -0.47 & \text{for } h=\pm 2 \\ 0 & \text{for } |h|>2 \end{cases}$$

ii) $X_t = Z_t + 0.3 Z_{t-1} - 0.4 Z_{t-2}$

$$\gamma(0) = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

$$= 1 + (0.3)^2 + (-0.4)^2$$

$$= 1.25$$

$$\gamma(1) = (\theta_1 + \theta_1\theta_2)\sigma^2$$

$$= 0.3 + (0.3) \times (0.4)$$

$$= 0.18$$

$\gamma(2) = \theta_2\sigma^2 = -0.4$

$$\therefore \gamma(h) = \begin{cases} 1.25 & \text{for } h=0 \\ 0.18 & \text{for } h=\pm 1 \\ -0.4 & \text{for } h=\pm 2 \\ 0 & \text{for } |h|>2 \end{cases}$$

$$\rho(h) = \begin{cases} 1 & \text{for } h=0 \\ 0.14 & \text{for } h=\pm 1 \\ -0.32 & \text{for } h=\pm 2 \\ 0 & \text{for } |h|>2 \end{cases}$$

Note:

MA(1)

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & ; h=0 \\ \theta\sigma^2 & ; h=\pm 1 \\ 0 & ; |h|>1 \end{cases}$$

Q10. Express an ARMA (p, q) process in terms of backward shift operator. How could we select the order of an ARMA process?

Solution:

$\phi(B)X_t = \theta(B)Z_t$; $\{Z_t\} \sim WN(0, \sigma^2)$.

B is the backward shift operator.

Selecting the order of an ARMA process:

Use information criteria like the Akaike information criterion (AIC) or the Bayesian information criterion (BIC) to compare different ARMA model with different orders.

- Fit ARMA models with various combinations of p and q.
- Calculate the AIC or BIC for each model.
- Select the model with lowest AIC or BIC as the fitting model.
- After selecting an order, perform diagnostic checks on the residuals of the fitted ARMA model to ensure that the model captures the underlying pattern in the data.

Q.11. Determine whether the following ARMA process is causal and/or invertible.

i) $X_t - 1.9X_{t-2} = Z_t + 0.8Z_{t-1} + 0.7Z_{t-2}$

ii) $X_t + 1.3X_{t-2} = Z_t + 0.3Z_{t-1} - 1.2Z_{t-2}$

where $\{Z_t\} \sim \cancel{WN(0, \sigma^2)}$ $WN(0, 1)$

Solution(i):

$\phi(z) = 1 - 1.9 z^2$

$z = \pm 0.725$

So, it is not causal.

$\theta(z) = 1 + 0.8z + 0.7z^2$

$$\Rightarrow z = \frac{-0.8 \pm \sqrt{(0.8)^2 - 4 \cdot 1 \cdot (0.7)}}{2 \times 0.7}$$

$$= -0.8 \pm \sqrt{(0.8)^2 - 4 \cdot}$$

$$= \frac{-0.8 \pm \sqrt{-2.16}}{1.4}$$

$$= \frac{-0.8 \pm 1.47i}{1.4}$$

$$z = \frac{-0.8 \pm i\sqrt{2.16}}{1.4}$$

and $|z|^2 = [(-0.8)^2 + 2.16] / (1.4)^2$

$= 1.429 > 1$

which implies the process is invertible.

ii) $X_t + 1.3 X_{t-2} = Z_t + 0.3 Z_{t-1} - 1.2 Z_{t-2}$

$\phi(z) = 1 + 1.3 z^2$

$z = 0.877$ or $z = -0.877$

So, the process is not causal

$\theta(z) = 1 + 0.3z - 1.2z^2$

$$\Rightarrow z = \frac{-0.3 \pm \sqrt{(0.3)^2 - 4 \cdot (-1.2) \cdot 1}}{2 \cdot (-1.2)}$$

$\therefore z = 1.046$ or $z = -0.79$

~~So, the process, is not causal~~

~~$\theta(z) = 1 + 0.3z - 1.2z^2$~~

So the process is not invertible.

Q12. What do you mean by AR(p) and MA(q) process. State the condition under which ARMA (1, 1) has unique stationary solution and hence find it.

Answer:

AR(P): While means that the process is autoregressive process of order p

Let, $\{X_t\}$ is said to be AR(P) process if

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\phi_1, \phi_2, \ldots, \phi_p$ are constants.

MA(q): which means that the process is moving average process of order q.

Let, $\{X_t\}$ is said to be a MA(q) process if

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \dots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \theta_2, \dots, \theta_q$ are constants.

ARMA(1,1)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1} \quad \dots (i)$$

- A stationary solution of the ARMA (1,1) equations exists if and only if $\phi \neq \pm 1$
- If $|\phi| < 1$, then the unique stationary solution is given by (ii). In this case, we say that $\{X_t\}$ is causal or a causal function of $\{Z_t\}$, since $\{X_t\}$ can be expressed in terms of the current and past values of Z_s, $s \le t$

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j} \quad \dots (ii)$$

- If $|\phi| > 1$, then the unique stationary solution is given by (iii). The solution is non-causal, Since X_t is then a function of z_s ; $s \geq t$

$$X_t = \theta\phi^{-1} Z_t - (\phi+\theta) \sum_{j=1}^{\infty} \phi^{-j-1} Z_{t+j}$$

Q.13: Define spectral decomposition, spectral analysis and spectral density of a time series.

Answer:

Spectral decomposition: The spectral decomposition is an analogue for stationary process of the more familiar Fourier representation of deterministic function. It decomposes a time series $\{X_t\}$ into a sum of sinusoidal components with uncorrelated random coefficients.

Spectral analysis: The analysis of stationary processes by means of their spectral representation is often referred to as the frequency domain analysis of time series or spectral analysis.

Spectral density: Suppose that $\{X_t\}$ is a zero mean stationary time series with autocovariance function $\gamma(\cdot)$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. The spectral density of $\{X_t\}$ is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h), \quad -\infty < \lambda < \infty \qquad \text{—(i)}$$

where,

$e^{i\lambda} = \cos(\lambda) + i\sin(\lambda)$ and $i = \sqrt{-1}$

The summability of $|\gamma(\cdot)|$ towards $\{X_t\}$ in equation (i) converges absolutely as

$$|e^{-ih\lambda}|^2 = \cos^2(h\lambda) + \sin^2(h\lambda) = 1$$

Q14. State and prove the basic properties of spectral densities.

Answer:

Properties:

a) f is even, i.e., $f(\lambda) = f(-\lambda)$,

b) $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$, and

c) $\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda)\, d\lambda$

Proof (a): since $\sin(\cdot)$ is a odd function and $\cos(\cdot)$ and $\gamma(\cdot)$ are even functions, we have

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} (\cos(h\lambda) - i\sin(h\lambda))\,\gamma(h)$$

$$= \frac{1}{2\pi} \sum_{h-\infty}^{\infty} \cos(-h\lambda)\,\gamma(h) + 0$$

$$\therefore f(\lambda) = f(-\lambda)$$

Proof (b): For each positive integer N define

$$f_N(\lambda) = \frac{1}{2\pi N} E\left(\left| \sum_{n=1}^{N} X_n e^{-in\lambda} \right|^2 \right)$$

$$= \frac{1}{2\pi N} E\left(\sum_{n=1}^{N} X_n e^{-in\lambda} \sum_{s=1}^{N} X_s e^{is\lambda} \right)$$

$$f_N(\lambda) = \frac{1}{2\pi N} \sum_{|h|<N} (N - |h|) e^{-ih\lambda} \gamma(h)$$

Clearly, the function f_N is nonnegative for each N, and since $f_N(\lambda) \rightarrow \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{ih\lambda} \gamma(h) = f(\lambda)$

as $N \rightarrow \infty$, f must also be nonnegative.

Proof (c): $$\int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{i(k-h)\lambda} \gamma(h)\, d\lambda$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \int_{-\pi}^{\pi} e^{i(k-h)\lambda}\, d\lambda$$

$$= \gamma(k)$$

Q15. How could you use the spectral density to estimate the autocovariance function (ACVF) of a time series process?

Solution:

We know.

A function f is the spectral density of a stationary time series $\{X_t\}$ with ACVF $\gamma(\cdot)$ if

(i) $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$, and

(ii) $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$ for all integers h.

- Spectral representation of ACVF

$\gamma(h)$ is a ACVF of $\{X_t\}$ if there exists a non decreasing bounded function F on $[-\pi, \pi]$ with $F(-\pi) = 0$

That is, $\gamma(h) = \int_{(-\pi,\pi]} e^{ih\lambda}\, dF(\lambda)$ for all integers

F is called the spectral distribution function of $\gamma(\cdot)$. $F(\lambda) = \int_{-\pi}^{\lambda} f(y)\, dy$ for all $\lambda \in [-\pi, \pi]$

Q16. Check whether the following function is the ACVF of a stationary process;

$$\gamma(h) = \begin{cases} 1 & \text{if } h=0 \\ -0.5 & \text{if } h=\pm 2 \\ -0.25 & \text{if } h=\pm 3 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

We have that $\gamma(h)$ is an even function. Since $\gamma(h) = \gamma(-h)$ and

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-3}^{3} e^{-ih\lambda} \gamma(h)$$

$$= \frac{1}{2\pi} \left[-0.25 e^{i3\lambda} - 0.5 e^{i2\lambda} + 1 - 0.5 e^{-i2\lambda} - 0.25 e^{-i3\lambda} \right]$$

$$= \frac{1}{2\pi} \left[1 - 0.25 \left(e^{i3\lambda} + e^{-i3\lambda} \right) - 0.5 \left(e^{i2\lambda} + e^{-i2\lambda} \right) \right]$$

$$= \frac{1}{2\pi} \left[1 - 0.25 \times 2\cos(3\lambda) - 0.5 \times 2\cos(2\lambda) \right]$$

$$= \frac{1}{2\pi} \left[1 - 0.5\cos(3\lambda) - \cos(2\lambda) \right]$$

which indicates that $f(\lambda)$ is not always non-negative on $\lambda \in (-\pi, \pi]$. For instance,

$$f(0) = -\frac{1}{4\pi}$$

Hence, $\gamma(h)$ is not an ACVF for stationary time series.

Q17. Find the spectral density of an MA(1) process $X_t = Z_t - 0.4 Z_{t-1}$, where $\{Z_t\} \sim WN(0,1)$. Hence, find the frequency spectrum of that process and comment.

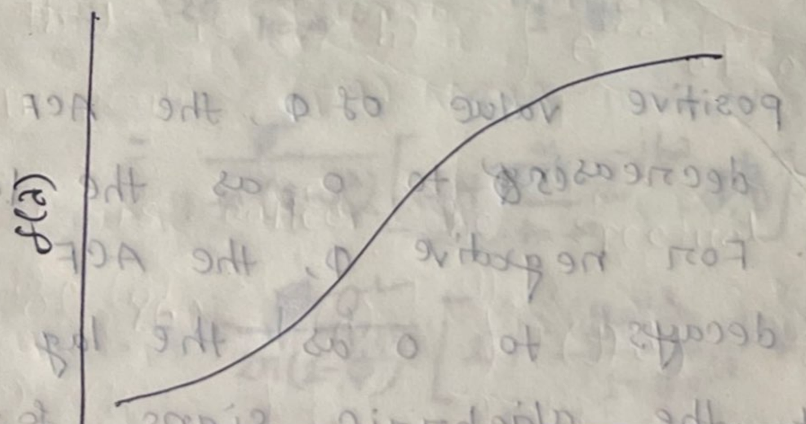
Solution:

We know that, the spectral density of an MA(1) process is,

$$f(\lambda) = \frac{\sigma^2}{2\pi}(1 + 2\theta \cos\lambda + \theta^2)$$

$$\Rightarrow f(\lambda) = \frac{1}{2\pi}(1 + 2\times(-0.4)\cos\lambda + (-0.4)^2)$$

$$\therefore f(\lambda) = \frac{1}{2\pi}(1 - 0.8\cos\lambda + 0.16)$$



The spectral density $f(\lambda)$, $0 \leq \lambda \leq \pi$, of $X_t = Z_t - 0.4 Z_{t-1}$, where $\{Z_t\} \sim WN(0, \sigma^2)$

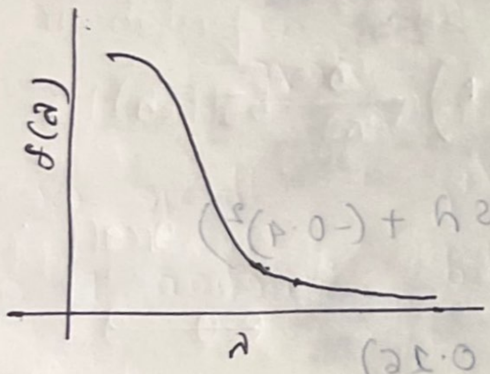
(slowly increasing function)

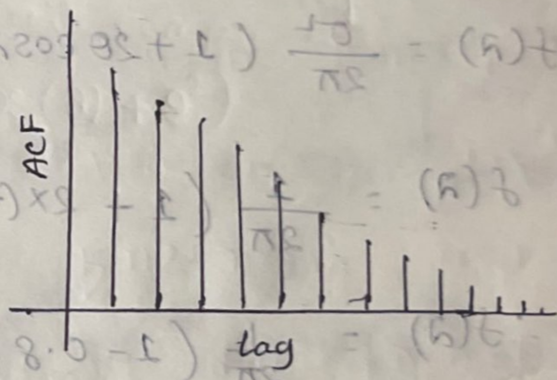
Q18. Under usual notations, write down the AR(1) process if $\phi = 0.5$.

Solution:

AR(1): $X_t = \phi X_{t-1} + Z_t$

$\therefore X_t = 0.5 X_{t-1} + Z_t$





$\rho(h) = \phi^h$

So, for a positive value of ϕ, the ACF exponentially decreasing to 0 as the lag h increases. For negative ϕ, the ACF also exponentially decays to 0 as the lag h increases, but the algebraic signs for the autocorrelations ~~alternative~~ alternate between positive and negative.

Q19. Why ~~Furier~~ Fourier transformation is used in spectral analysis?

Answer:

Because it allows us to decompose a complex valued function into its frequency components.

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} \, dx$$

Q20. Find the spectral density of an AR(1) process.

$X_t = \phi X_{t-1} + Z_t$

Solution:

Here, we know the spectral density:

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)$$

For AR(1)

ACVF

$$\gamma(h) = \frac{\sigma^2 \phi^h}{1-\phi^2}$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \cdot \frac{\sigma^2 \phi^h}{1-\phi^2}$$

$$= \frac{\sigma^2}{2\pi(1-\phi^2)} \left[1 + \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \phi^h \right]$$

$$= \frac{\sigma^2}{2\pi(1-\phi^2)} \left[1 + \frac{\phi e^{i\lambda}}{1-\phi e^{i\lambda}} + \frac{\phi e^{-i\lambda}}{1-\phi e^{-i\lambda}} \right]$$

$$= \frac{\sigma^2}{2\pi} \left[1 - 2\phi \cos\lambda + \phi^2 \right]^{-1}$$

Q21. Find the spectral density of MA(1) process:

$$Z_t = X_t + \theta X_{t-1}$$

Solution:

Here we know, the spectral density:

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma(h)$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \cdot \frac{\sigma^2 \theta}{1+\theta^2}$$

$$= \frac{\sigma^2}{2\pi} \left(1+\theta^2 + \theta\left(e^{-i\lambda} + e^{i\lambda}\right)\right)$$

For MA(1) ACVF $\gamma(h) = \frac{\sigma^2 \theta}{1+\theta^2}$

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left(1 + 2\theta\cos(\lambda) + \theta^2\right)$$

~~Here~~,

$\cancel{\theta = 0.9}$

Q22. Find the spectral density of an ARMA process $\{X_t\}$ for ARMA (P, q)

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

$$\phi(B) X_t = \theta(B) Z_t$$ where $Z_t \sim WN(0, \sigma^2)$

Solution:

ACVF $\gamma(h) = E(X_{t+h}, X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} |\gamma(h)|$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \left(\sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \right)$$

$$= \frac{\sigma^2 |\theta(e^{-i\lambda})|^2}{2\pi |\phi(e^{-i\lambda})|^2} \; ; \; -\pi \le \lambda \le \pi$$

$\phi(\cdot)$ and $\theta(\cdot)$ are p^{th} and q^{th} degree poly polynomials.

$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$

and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$

$B^j X_t = X_{t-j}$ and $B^j Z_t = Z_{t-j}$

$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ and $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$

Applying the function

$$\psi(e^{-i\lambda}) = \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})}$$

Q23. What is a seasonal ARIMA (SARIMA) model?

Answer:

- ARIMA is one of the most widely used forecasting methods for univariate time series component data forecasting.
- The method can handle data with a trend, it does not support time series with a seasonal component.

An extension to ARIMA that supports the direct modeling of the seasonal components of the series is called SARIMA.

Mathematical representation of ARIMA to SARIMA:

If d is a nonnegative integer, then $\{X_t\}$ is an ARIMA (p, d, q) process if $Y_t = (1-B)^d X_t$ is a causal ARMA (p, q) process.

That means,

$$\phi(B) Y_t = \phi(B)(1-B)^d X_t = \theta(B) Z_t, \quad \{Z_t\} \sim WN(0, \sigma^2)$$

- If we fit an ARMA (p, q) model $\phi(B) Y_t = \theta(B) Z_t$ to the different time series

$Y_t = (1-B^s)X_t$, then the model for the original series is, $\phi(B)(1-B^s)X_t = \theta(B)Z_t$

This is a special case of the general SARIMA model defined as follows:

- If d and D are non-negative integers, then $\{X_t\}$ is a seasonal ARIMA $(p,d,q)\times(P,D,Q)_s$ process with period s if the differenced series $Y_t = (1-B)^d(1-B^s)^D X_t$ is a causal ARMA process defined by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t, \quad \{Z_t\} \sim WN(0,\sigma^2)$$

where,

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \text{ and}$$

$$\Theta(z) = 1 + \Theta_1 z + \cdots + \Theta_Q z^Q$$

Q-24. When do you prefer ARIMA to ARMA model?

Answer:

We should prefer ARIMA models over ARMA models when dealing with non-stationary time series data to exhibit trends, seasonality or require differencing to achieve stationarity.

PACF of an AR(P)

$\alpha(P) = \phi_P$

and $\alpha(h) = 0$ for $h > P$

PACF of an MA (1)

$\alpha(h) = \phi_{hh} = \frac{-(-\theta)^h}{1+\theta^2+\cdots+\theta^{2h}}$

Q25. In preliminary estimation of an ARMA process which method is useful for pure auto regressive model and why?

Answer:

For a pure autoregressive model, the PACF is useful. Because, it helps identify the order of the AR process by capturing direct influences from past values while eliminating shorter lag influences.

Note:

If there are significant spikers at mutiple lags, it suggest an MA process (ACF)

Q26. What is meant by diagnostic checking of a time series model? What are the possible ways for doing it?

Answer:

The goodness of fit of a statistical model to a set of data is judged by comparing the observed values with the corresponding predicted values obtained from the fitted model.

If the fitted model ~~mod~~ is appropriate, then the residuals should behave in a manner that is consistent with the model.

The residuals are then.

$$\hat{W}_t = [X_t - \hat{X}_t(\hat{\phi}, \hat{\theta})] / [r_{t-1}(\hat{\phi}, \hat{\theta})]^{1/2}, \quad t = 1, \ldots, n$$

In particular, the sequence $\{W_t\}$ should be approximately

1) Uncorrelated if $\{Z_t\} \sim WN(0, \sigma^2)$
2) Independent if $\{Z_t\} \sim IID(0, \sigma^2)$
3) normally distributed if $\{Z_t\} \sim N(0, \sigma^2)$

The rescaled residuals

$\hat{R}_t = \hat{W}_t / \hat{\sigma}$

Possible ways:

- The graph of the rescaled residuals.
- The sample ACF of the residuals.
- Tests for randomness of the residuals.

Note: Recall testing estimated noise sequence.

Q-27. Discuss the process of estimating the parameters of an ARMA (p, q) process by the maximum likelihood method.

Answer:

The Gaussian likelihood for an ARMA process:

$$L(\phi, \theta, \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)^n r_0 \cdots r_{n-1}}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}\frac{(X_j - \hat{X}_j)^2}{r_{j-1}}\right\}$$

Maximum likelihood estimators:

$\hat{\sigma}^2 = n^{-1} S(\hat{\phi}, \hat{\theta})$

where,

$$S(\hat{\Phi}, \hat{\theta}) = \sum_{j=1}^{n} (X_j - \hat{X}_j)^2 / r_{j-1}$$

and $\hat{\Phi}$, $\hat{\theta}$ are the values of Φ, θ that minimize

$$l(\Phi, \theta) = \ln(n^{-1} S(\hat{\Phi}, \hat{\theta})) + n^{-1} \sum_{j=1}^{n} \ln r_{j-1}$$

Q28. Differentiate between ARMA and SARIMA model.

Answer:

ARMA	SARIMA
1. ARMA takes into account the past values (AR, MA) and predict future values on that	1. SARIMA similarly uses past values but also takes into account any seasonal patterns.
2. It is denoted by ARMA (p, q); where p is the order of AR and q is the order of MA.	2. It is denoted by ARIMA $(p, d, q) \times (P, D, Q)_s$; where p, d, q are three trend elements and P, D, Q and S are four seasonal elements.
3. It is a univariate time series.	3. It is a multivariate time series.

ARMA	SARIMA
4. It is used for estimating trend.	4. It is used for estimating trend and adjusting seasonality
5. It has 2 parameters.	5. It has 6 parameters.
6. The interse of complexity or parameter is less than SARIMA	6. Interse of complexity and parameter is more than ARMA

~~Q.29. Differentiate between~~

Q.29. Distinguish between an ARIMA and a SARIMA process:

Answer:

ARIMA	SARIMA
1. Composed of three components: AR (autoregressive) I (Integrated), MA (Moving average)	2. Extends ARIMA by adding seasonal components (P, D, Q, S) for seasonal autoregressive, seasonal differencing, seasonal moving average and the length of the seasonal period

ARIMA	SARIMA
2. No seasonal components in the model	2. Has seasonal components.
3. Differencing (d) is used to make the series stationary.	3. Seasonal differencing (D) is added to ARIMA to handle seasonal patterns in addition to non-seasonal differencing.
4. Suitable for non-seasonal time series.	4. Suitable for time series data with seasonal effects.
5. Does not account for seasonal trends directly 6. Forcasts are based on non-seasonal trends and patterns.	5. Specifically designed to handle seasonal patterns in the data 6. Forcasts ~~co~~ incorporate both seasonal and non-seasonal components.
7. Example: Used for time series without periodic patterns (e.g., temperature data in a non-repetitive setting.	7. Example: Used for data exhibiting seasonal behavior (e.g., monthly electricity consumption)

Q30. Derive the autocovariance function for an ARMA (1, 1) process.

Solution:

For ACVF we have

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1} \qquad \{Z_t\} \sim WN(0, \sigma^2)$$

with $|\phi| < 1$ is given by

$$\gamma(0) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2$$

$$= \sigma^2 \cancel{\sum} \left[1 + (\theta + \phi)^2 \sum_{j=0}^{\infty} \phi^{2j} \right]$$

$$= \sigma^2 \left[1 + \frac{(\theta + \phi)^2}{1 - \phi^2} \right]$$

$$\gamma(1) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j-1} \psi_j$$

$$= \sigma^2 \left[\theta + \phi + (\theta + \phi)^2 \phi \sum_{j=0}^{\infty} \phi^{2j} \right]$$

$$= \sigma^2 \left[\theta + \phi + \frac{(\theta + \phi)^2 \phi}{1 - \phi^2} \right]$$

and

$$\gamma(h) = \phi^{h-1} \gamma(1). \qquad h \geq 2$$

Q31. Define (i) memory of a time series; (ii) memory ~~shorting~~ shortening; (iii) subset of autoregression.

Answer:

(i) Memory of a time series:

The memory of a time series is a measure of how much past values influence the future values in a given variable. A time series with a strong memory has a past that is very indicative of the future. This memory can be short memory or long memory.

(ii) Memory shortening:

Memory shortening is when the influence of past ~~events~~ values on future ~~events~~ values decreases more quickly over time. This ~~can~~ Short-term memory is the ability to store and access a small amount of information for a short time. When short-term memories are not actively maintained, they can last only seconds.

(iii) Subset of autoregression:

A subset of autoregression in time series refers to using only a part of previous data points to predict the current or future values in a time series. AR is a model where the current value is based on its previous values. A subset means we are only considering a limited number of past data points instead of all previous ones.

Q32: What is prediction operator?

Answer:

A prediction operator is a function or mechanism used to generate predictions or forecasts based on input data and a trained model. It maps the ~~pre~~ input variables to predicted outcomes or target values.

Periodogram:

The periodogram of ~~fx~~ $\{x_1, \dots, x_n\}$ is the function

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^{n} x_t e^{-it\lambda} \right|^2$$

Q33. What is a partial autocorrelation function?

Answer:

The partial autocorrelation function (PACF) of an ARMA process $\{X_t\}$ is the function $\alpha(\cdot)$ defined by the equations

$$\alpha(0) = 1$$

and $\alpha(h) = \phi_{hh}, \quad h \geq 1$

where ϕ_{hh} is the last component of $\phi_h = \Gamma_h^{-1} \gamma_h$,

$\Gamma_h = [\gamma(i-j)]_{i,j=1}^{h}$, and $\gamma_h = [\gamma(1), \gamma(2), \ldots, \gamma(h)]'$

For any set of observations $\{X_1, \ldots, X_n\}$ with $x_i \neq x_j$ for some i and j, the sample PACF $\hat{\alpha}(h)$ is given by

$\hat{\alpha}(0) = 1$ and

$\hat{\alpha}(h) = \hat{\phi}_{hh}, \quad h \geq 1$

where $\hat{\phi}_{hh}$ is the last component of

$$\hat{\phi}_h = \hat{\Gamma}_h^{-1} \hat{\gamma}_h.$$

Q34. Define an ARMA(2,1) process.

Answer:

An ARMA(2,1) process is a time series model that combines two components: An (AR) part of order 2 and moving Average (MA) part of order 1. It is used to model time series data where the currend value depends on both past values of the series and past error terms.

An ARMA(2,1) process is defined by the following equation:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \theta_1 \varepsilon_{t+1} + \varepsilon_t$$

where

y_t is the value of the time series at time t

ϕ_1 and ϕ_2 are the coefficients of the AR(2) component

θ_1 is the coefficient of the MA(1) component.

ε_t is the white noise (error) term at time t.

Q35. "Not all autocovariance functions have a spectral density" - justify your answer with a suitable example.

Answer:

"Not all autocovariance functions have a spectral density" because the autocovariance function must be positive semi-definite for the spectral density to exist. If the autocovariance function is not positive semi-definite, a valid spectral density cannot be defined.

Example:

An example of an autocovariance function that does not have a spectral density is the following:

$$\gamma(\tau) = \begin{cases} 1, & \text{if } \tau = 0 \\ -1, & \text{if } \tau = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

This autocovariance function is not ~~possible~~ positive semi-definite, which means it doesn't satisfy the necessary conditions for the existence of a spectral density. For some lags, the values are negative, leading to a covariance matrix with negative eigenvalues, which breaks the positive semi-definiteness requirement.

Q36. Why do we need to measure memory?

Answer:

1) In memory measure, we are basically talking about how correlated different points are on the timeline.

2) If we increase the temporal distance between two points, how strongly will they continue to correlated we know.

3) Measuring memory has a huge impact on the predictability of the data.

4) If we want to do things like prediction or forecasting, we need to estimate the memory or else we will get very spurious results.

5) Long ~~Short~~ memory process are fascinating because the past strongly influences the present and the future.

(6) Short memory processes don't lend themselves nicely to analysis because the

past doesn't impact the future that much.

7) It would be very useful to estimate the long-term memory of a variable.

Forecasting techniques:

1. ARAR algorithm.
2. Holt-Winters algorithm.
3. The Holt-Winters seasonal (HWS) algorithm.

Q37. Define ARAR algorithm.

Answer:

The ARAR algorithm is basically the process that applied shortening transformation and fitting the ~~autog~~ autoregressive (AR) model to the transformed data. It is used to predict the future data from existing sequence data.

Q 38. Define Holt-Winter algorithm.

Answer:

Holt-Winters is a model of time series behavior. Forecasting always requires a model, and Holt-Winters is a way to model three aspects of the time series; a typical value (average), a slope (trend) over time and a cyclical repeating pattern (seasonality)

• Exponential Smoothing:

Simple exponential smoothing as the name suggest is used for forecasting when the data set has no trend or seasonality.

Holt's Smoothing method:

Holt's smoothing the technique, also known as linear exponential smoothing, is a widely known smoothing model for forecasting data that has a trend.

Winter's smoothing method:

Winter's smoothing technique allows us to include seasonality while making the prediction along with the trend.