Example 1: Suppose $[X(t), t \in T]$ be a stochastic process where $P_r[X(t)=n] = \frac{e^{-at}(at)^n}{n!}$; $n = 0, 1, 2, \cdots, a > 0$. Is this process stationary?

Solution:

We know that the mean function.

$$m(t) = E[X(t)]$$

$$= E[X(t)=n]$$

$$= \sum_{n=0}^{\infty} n P[X(t)=n]$$

$$= \sum_{n=0}^{\infty} n \cdot \frac{e^{-at}(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-at}(at)^n}{(n-1)!}$$

$$= e^{-at}(at) \sum_{n=1}^{\infty} \frac{(at)^{n-1}}{(n-1)!}$$

$$= e^{-at}(at)\left[1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots\right]$$

$$= e^{-at}(at)e^{at}$$

$$\therefore m(t) = at$$

We know

$$V[X(t)] = E[\{X(t)\}^2] - [E\{X(t)\}]^2$$

Now,

$$E[\{X(t)\}^2] = \sum_{n=0}^{\infty} n^2 \frac{e^{-at}(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-at}(at)^n}{n!} + \sum_{n=0}^{\infty} n \frac{e^{-at}(at)^n}{n!}$$

$$= (at)^2 e^{-at} \sum_{n=2}^{\infty} \frac{(at)^{n-2}}{(n-2)!} + (at) e^{-at} \sum_{n=1}^{\infty} \frac{(at)^{n-1}}{(n-1)!}$$

$$= (at)^2 e^{-at} \cdot e^{at} + at$$

$$\therefore E[\{X(t)\}^2] = (at)^2 + at$$

Now,

$$V[X(t)] = E[\{X(t)\}^2] - [E\{X(t)\}]^2$$

$$= (at)^2 + at - (at)^2$$

$$\therefore V[X(t)] = at$$

Since the mean and variance functions of this stochastic process are dependent on time t, so the stochastic process $[X(t), t \in T]$ is non-stationary or evolutionary process

CamScanner

Example 2: Consider the process $X(t) = A_1 + A_2 t$ where A_1 and A_2 are independent random variables with $E(A_i) = a_i$, $V(A_i) = \sigma_i^2$; $i = 1, 2$. Test the stationarity.

Solution:

We know that the mean function is

$$m(t) = E[X(t)]$$

$$= E[A_1 + A_2 t]$$

$$= a_1 + a_2 t$$

$$E[X(t)]^2 = E[(A_1 + A_2 t)^2]$$

$$= E[A_1^2 + A_2^2 t^2 + 2 A_1 A_2 t]$$

$$= E[A_1^2] + t^2 E[A_2^2] + 2t\, E[A_1 A_2]$$

$$= \sigma_1^2 + a_1^2 + t^2(\sigma_2^2 + a_2^2) + 2t\, a_1 a_2$$

$$\therefore V[X(t)] = E[\{X(t)\}^2] - [E\{X(t)\}]^2$$

$$= \sigma_1^2 + a_1^2 + t^2(\sigma_2^2 + a_2^2) + 2t a_1 a_2 - (a_1 + a_2 t)^2$$

$$= \sigma_1^2 + a_1^2 + t^2\sigma_2^2 + t^2 a_2^2 + 2t a_1 a_2 - a_1^2 - 2a_1 a_2 t - a_2^2 t^2$$

CamScanner

$\therefore V[X(t)] = \sigma_1^2 + t^2\sigma_2^2$

$$E[X(t)\cdot X(s)] = E[(A_1+A_2t)(A_1+A_2s)]$$

$$= E[A_1^2 + A_1A_2s + A_1A_2t + A_2^2ts]$$

$$= E[A_1^2 + A_2^2ts + A_1A_2(t+s)]$$

$$= \sigma_1^2 + a_1^2 + ts(\sigma_2^2 + a_2^2) + (t+s)a_1a_2$$

$$C(s,t) = \text{cov}[X(s), X(t)]$$

$$= E[X(s).X(t)] - E[X(s)]E[X(t)]$$

$$= \sigma_1^2 + a_1^2 + ts(\sigma_2^2 + a_2^2) + a_1a_2(t+s) -$$

$$(a_1+a_2s)(a_1+a_2t)$$

$$= \sigma_1^2 + a_1^2 + ts\sigma_2^2 + tsa_2^2 + a_1a_2t +$$

$$a_1a_2s - a_1^2 - a_1a_2t - a_1a_2s - a_2^2ts$$

$$= \sigma_1^2 + ts\sigma_2^2$$

$$\therefore C(s,t) \neq f(t-s)$$

CamScanner

Since the mean and variance function as well as the covariance function of this stochastic process are dependent on time t, so the stochastic process $[X(t), t \in T]$ is non-stationary / evolutionary process.

Example 3: Consider the process $X(t) = A\cos\omega t + B\sin\omega t$ where A, B uncorrelated random variables each with are mean 0 and variance 1 and ω is a positive constant. Show that the process is covariance stationary.

Solution:

Given, $E(A) = 0$, $E(B) = 0$, $E(AB) = 0$

$V(A) = 1$, $V(B) = 1$

$E(A^2) = V(A) + \{E(A)\}^2$

$= 1 + 0$

$\therefore E(A^2) = 1$

Now the mean function

$m(t) = E[X(t)]$

$= \cos\omega t \; E(A) + \sin\omega t \; E(A) = 0$

CamScanner

$$E[\{X(t)\}^2] = E[(A\cos\omega t + B\sin\omega t)^2]$$

$$= E[A^2\cos^2\omega t + B^2\sin^2\omega t + 2AB\cos\omega t \cdot \sin\omega t]$$

$$= \cos^2\omega t\, E(A^2) + \sin^2\omega t\, E(B^2) +$$

$$2\cos\omega t \cdot \sin\omega t\ E(AB)$$

$$= \cos^2\omega t + \sin^2\omega t$$

$$\therefore E[\{X(t)\}^2] = 1$$

$$\therefore V[X(t)] = E[\{X(t)\}^2] - \{E[X(t)]\}^2$$

$$= 1 - 0$$

$$\therefore V[X(t)] = 1$$

$$E[X(t) \cdot X(s)] = E[(A\cos\omega t + B\sin\omega t) \cdot (A\cos\omega s + B\sin\omega s)]$$

$$= E[A^2\cos\omega t \cdot \cos\omega s + AB\cos\omega t \cdot \sin\omega s +$$

$$AB\sin\omega t \cdot \cos\omega s + B^2\sin\omega t \cdot \sin\omega s]$$

$$= \cos\omega t \cdot \cos\omega s\, E(A^2) + \cos\omega t \cdot \sin\omega s\, E(AB)$$

$$+ \sin\omega t \cdot \cos\omega s\, E(AB) + \sin\omega t \cdot \sin\omega s\, E(B^2)$$

$$= \cos\omega t \cdot \cos\omega s + \sin\omega t \cdot \sin\omega s$$

$$\therefore E[X(t) \cdot X(s)] = \cos(s-t)\omega \quad [\because \cos(A-B) \text{ formula}]$$

$\therefore C(s,t) = \text{cov}[X(t), X(s)]$

$= E[X(t) \cdot X(s)] - E[X(t)] \cdot E[X(s)]$

$= \cos(s-t)\, co$

Here the first two moments are finite and the covariance function is a function of $(s-t)$. Thus, the process is covariance stationary.

Example 4: Consider the process $[X(t), t \in T]$ whose probability distribution, under a certain condition, is given by

$$Pr[X(t) = n] = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n = 1, 2, \ldots$$

$$= \frac{at}{1+at}, \quad n = 0$$

Test the ~~stationary~~ stationarity of the process.

Solution: We have the mean function,

$m(t) = E[X(t)]$

$= \sum_{n=0}^{\infty} n \cdot Pr(X(t) = n)$

CamScanner

$$= 0 \times \cancel{(at)^{-}}$$

$$= 0 \times \frac{(at)^{-1}}{1+at} + \sum_{n=1}^{\infty} n \cdot \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\cdot\frac{at}{1+at} + 3\cdot\left(\frac{at}{1+at}\right)^2 + \cdots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2} \qquad \left[\because 1+2x+3x^2 = (1-x)^{-2}\right]$$

$$= \frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}$$

$$m(t) = 1$$

and,

$$E\left[\{X(t)\}^2\right] = \sum_{n=0}^{\infty} n^2\, P[X(t) = n]$$

$$= \sum_{n=9}^{\infty} n^2 \cdot \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=9}^{\infty} n\cdot(n-1)\, \frac{(at)^{n-1}}{(1+at)^{n+1}} + \sum_{n=1}^{\infty} n \cdot \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{at}{(1+at)^3} \sum_{n=2}^{\infty} n(n-1)\left(\frac{at}{1+at}\right)^{n-2} + 1$$

$$= \frac{at}{(1+at)^3}\left[2 + 3\cdot 2\,\frac{at}{1+at} + 4\cdot 3\left(\frac{at}{1+at}\right)^2 + 5\cdot 4\left(\frac{at}{1+at}\right)^3 + \cdots\right] + 1$$

$$= \frac{2at}{(1+at)^3}\left[1 + 3\,\frac{at}{1+at} + 6\left(\frac{at}{1+at}\right)^2 + 10\left(\frac{at}{1+at}\right)^3 + \cdots\right] + 1$$

$$= \frac{2at}{(1+at)^3}\left[1 - \frac{at}{1+at}\right]^{-3} + 1 \qquad \left[\because 1+3x+6x+10x = (1-x)^{-3}\right]$$

$$= \frac{2at}{(1+at)^3}\left[\frac{1}{1+at}\right]^{-3} + 1$$

$\therefore \cancel{V[X(t)]} = \cancel{2at+1}$

$E[\{X(t)\}^2] = 2at+1$

$\therefore V[X(t)] = 2at + 1 - 1 = 2at$

Here, the first moment is constant but the second moment and (variance) increases with t. thus the process $[X(t), t \in T]$ is evolutionary.

Martingales process

Definition: A discrete parameter stochastic process $\{X_n\}_{n=0}^{\infty}$ is called a Martingale or (a Martingale process) if,

(i) $E[|X_n|] < \infty$ and

(ii) $E[X_{n+1} | X_n, X_{n-1}, \dots X_0] = X_n$

Example: Let $\{Z_i\}$; $i = 1, 2, \dots$ be a sequence of i.i.d random variables with mean 0 and let $X_n = \sum_{i=1}^{n} Z_i$, then show that $\{X_n\}_{n=1}^{\infty}$ is a martingale.

Solution: We have

$$E\{Z_i\} = 0$$

$$X_n = Z_1 + Z_2 + \dots + Z_n \quad \text{——— (1)}$$

$$\Rightarrow E(X_n) = E\left\{\sum_{i=1}^{n} Z_i\right\} = \sum_{i=1}^{n} E\{Z_i\} = 0 \text{; since } Z_i\text{'s are i.i.d and with mean } 0$$

CS CamScanner

Now, we have,

$$X_{n+1} = Z_1 + Z_2 + \cdots + Z_n + Z_{n+1}$$

$$= X_n + Z_{n+1} \quad \text{------ (1)}$$

So that,

$$E\{X_{n+1} \mid X_n, X_{n-1}, \ldots, X_1\} = E\{(X_n + Z_{n+1}) \mid X_n, X_{n-1}, \ldots, X_1\}$$

$$(1) \longrightarrow \quad = E\{X_n \mid X_n, X_{n-1}, \ldots, X_1\} +$$

$$E\{Z_{n+1} \mid X_n, X_{n-1}, \ldots, X_1\}$$

$$= X_n + E\{Z_{n+1}\}$$

$$= X_n \quad ; \quad \text{Since } E\{Z_{n+1}\} = 0$$

So, the process is martingale process because it satisfies the two condition of martingale process

Example:

Let $\{Z_i\}$; $i = 1, 2, \dots$ be a sequence of iid. random variables with $E\{Z_i\} = 1$ and let $X_n = \prod_{i=1}^{n} Z_i$, then show that $\{X_n\}_{n=1}^{\infty}$ is a martingale.

Solution: Here, given that

$$E\{Z_i\} = 1 ; \quad X_n = \prod_{i=1}^{n} Z_i \quad \text{———— (1)}$$

$$E(X_n) = E\left\{\prod_{i=1}^{n} Z_i\right\} = \prod_{i=1}^{n} E\{Z_i\} = 1 ; \quad Z_i\text{'s are independent}$$

Again, we have,

$$X_n = Z_1 \cdot Z_2 \cdot \dots \cdot Z_n$$

$$X_{n+1} = Z_1 \cdot Z_2 \cdot \dots Z_n \cdot Z_{n+1}$$

$$= X_n \cdot Z_{n+1} \quad [\text{using equation (1)}]$$

So that,

$$E\{X_{n+1} | X_n, X_{n-1}, \dots, X_1\} = E\{(X_n \cdot Z_{n+1}) | X_n, X_{n-1}, \dots, X_1\}$$

$$= E\{X_n | X_n, X_{n-1}, \dots, X_1\} \cdot E\{Z_{n+1} | X_n, X_{n-1}, \dots, X_1\}$$

$= X_n \cdot E\{Z_{n+1}\}$: Since Z_{i+1} and $X_n, X_{n-1}, \dots, X_1$ are independent

~~X_n~~

$= X_n$; Since $E\{Z_{n+1}\} = 1$

So, the process is martingale process because it satisfy the two condition of martingale process.

Note: The mean of martingale process is constant.

We know that,

$$X_n = \sum_{i=1}^{n} Z_i$$

$$X_{n+1} = X_n + Z_{n+1}$$

$$E\{X_{n+1}\} = E\{X_n\} + E\{Z_{n+1}\}$$

$$= E\{X_n\} + 0 \quad ; \text{Since } E\{Z_{n+1}\} = 0$$

$$\therefore E\{X_{n+1}\} = E\{X_n\}$$

Thus, we have for $n = 0, 1, \ldots$

$$E\{X_n\} = E\{X_{n-1}\} = E\{X_{n-2}\} = \cdots = E\{X_1\} = E\{X_0\}$$

CamScanner

~~**Example:** Let $\{Z_i\}$~~

Definition 2: Let $\{X_n\}_{n=0}^{\infty}$ and $\{Y_n\}_{n=0}^{\infty}$ be two discrete parameter stochastic processes. The process $\{X_n\}_{n=0}^{\infty}$ is said to be a martingale with respect to the process $\{Y_n\}_{n=0}^{\infty}$, if for $i = 0, 1, 2, \cdots$

(i) $E[|X_n|] < \infty$ and

(ii) $E[X_{n+1} | Y_n, Y_{n-1}, \cdots, Y_0] = X_n$

Example: Let $\{Z_i\}$; $i = 1, 2, \cdots$ be a sequence of i.i.d. random variables with mean 0 and let $X_n = \sum_{i=1}^{n} Z_i$, then show that $\{X_n\}_{n=1}^{\infty}$ is a martingale process with respect to $\{Z_n\}_{n=1}^{\infty}$

Solution:

We have,

$$E(X_n) = E\left\{\sum_{i=1}^{n} Z_i\right\} = \sum_{i=1}^{n} E(Z_i) = 0;$$ Since Z_i's are i.i.d and with mean 0

we have

$$X_n = Z_1 + Z_2 + \cdots + Z_n$$

$$\Rightarrow X_{n+1} = Z_1 + Z_2 + \cdots + Z_n + Z_{n+1} = X_n + Z_{n+1}$$

CamScanner

So that,

$$E\{X_{n+1} \mid Z_n, Z_{n-1}, \ldots Z_1\} = E\{\cancel{X_{n+1}}\ E\{(X_n + Z_{n+1}) \mid Z_n, Z_{n-1}, \ldots Z_1\}$$

$$= E\{X_n \mid Z_n, Z_{n-1}, \ldots, Z_1\} + E\{Z_{n+1} \mid Z_n, Z_{n-1}, \ldots Z_1\}$$

$$= X_n + E\{Z_{n+1}\}$$; Since Z_i's are independent

$$= X_n$$; Since $E\{Z_{n-1}\} = 0$

Since the given process satisfies the necessary two properties of martingale process, thus, we can say that $\{X_n\}_{n=1}^{\infty}$ is a martingale process with respect to $\{Z_n\}_{n=1}^{\infty}$

Sub-Martingale: **Definition 3:**

A stochastic process $\{X_n\}_{n=0}^{\infty}$ with $E[|X_n|] < \infty$ is said to be a sub-martingale, if

$$E\{X_{n+1} \mid X_n, X_{n-1}, \ldots, X_0\} \geq X_n \quad \text{--- (1)}$$

and is said to be a super-martingale if

$$E\{X_{n+1} \mid X_{n-1}, \ldots, X_0\} \leq X_n \quad \text{--- (2)}$$

If follows from (1) that, in case of a sum-martingale.

$$E\{X_{n+1}\} \geq E\{X_n\} \geq \cdots \geq E\{X_0\}$$

and in case of a super - martingale

$$E\{X_{n+1}\} \leq E\{X_n\} \leq \cdots \leq E\{X_0\}$$

Example: Suppose that the chance of rain tomorrow depends on today's weather conditions and not on yesterday's weather conditions. It is revealed that if it rains today, then it will rain tomorrow with probability 0.7 and if it does not rain today, then it will rain tomorrow with probability 0.4

(a) Is it a Markov chain? State the condition.

(b) Construct the TPM of the MC.

(c) What is the probability of raining on Tuesday given that it was not raining Monday?

(d) What is the probability of raining on Tuesday given that it was not raining Sunday?

Solution:

(a) Since the chance of rain ~~tho~~ tomorrow depends on today's weather conditions and not on yesterday's weather conditions, thus it is a problem of Markov chain.

(b) Suppose X_n represents the weather condition, where

$$X_n = \begin{cases} 0; & \text{raining} \\ 1; & \text{not raining} \end{cases}$$

Then $\{X_n; n \geq 0\}$ is a Markov chain with state space $S = \{0, 1\}$ and TPM

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.7 & 0.3 \\ 1 & 0.4 & 0.6 \end{array}$$

N.B

i, j		today, tomorrow
(0, 0)	=	rain, rain → 0.7
(0, 1)	=	rain, not " →
(1, 0)	=	not rain, rain → 0.
(1, 1)	=	not ", not "

(c) The probability of raining on Tuesday given that it was not raining Monday is $P_{10} = .4$.

This implies that if it was raining on Monday there is a 40% chance of raining on Tuesday.

(d) The probability of raining on Tuesday given that it was not raining sunday is $P_{10}^{(2)} = ??$

$$P^2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.61 & 0.39 \\ 1 & 0.52 & 0.48 \end{array}$$

this implies that if it was not raining on Sunday there is a 52% chance of raining on Tuesday.

CamScanner

Class Work (i & ii)

Solution of (i):

Suppose X_n represents the mood of a corporate president, where

$$X_n = \begin{cases} 0; & \text{cheerful} \\ 1; & \text{so-so} \\ 2; & \text{glum} \end{cases}$$

Then $\{X_n, n \geq 0\}$ is a Markov chain with state space $S = \{0, 1, 2\}$ and TPM will be

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

Solution of (ii):

The probability of being cheerful on Tuesday given that the president was glum on Sunday is $P_{20}^{(2)} = ?$

$$P^2 = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0.39 & 0.39 & 0.22 \\ 0.33 & 0.37 & 0.30 \\ 0.29 & 0.35 & 0.36 \end{bmatrix} \end{array}$$

This implies that the probability that the president will be in a cheerful mood on Tuesday, given that he was in a glum mood on Sunday is 0.~~29~~ w~~hich~~ 0.29. which means there is a 29% chance that his mood will be cheerful on Tuesday after two days.

Order of a Markov Chain:

A Markov chain $\{X_n : n \geq 0\}$ is said to be of order s ($s = 1, 2, \ldots$) if for all n,

$$P_r(X_n = j \mid X_{n-1} = i, X_{n-2} = h, \ldots, X_{n-s} = i_s, X_{n-(s+1)} = i_{s+1} \ldots)$$

$$= P_r(X_n = j \mid X_{n-1} = i, X_{n-2} = h, \ldots, X_{n-s} = i_s)$$

A Markov chain $\{X_n\}$ is said to be of order one (one simply a Markov chain) if

$$P_r(X_n = j \mid X_{n-1} = i, X_{n-2} = i_2, \ldots) = P_r(X_n = j \mid X_{n-1} = i)$$

$$= P_{ij}$$

A Markov chain $\{X_n\}$ is said to be of order two if

$$P_r(X_n = j \mid X_{n-1} = i, X_{n-2} = h, \ldots) = P_r(X_n = j \mid X_{n-1} = i, X_{n-2} = h)$$

Note: Normally this type of Markov chain does not exist.

A chain is said to be of order zero if $P_{ij} = P_j$ for all j. It is a memory less process which means that the observations are independent

Theorem: Show that all unconditional probabilities can be computed by conditioning on the Initial state.

Proof: ~~$P_{ij}^{(n)} = P_r[X_n = j \mid X_0 = i]$~~

$P_{ij}^{(n)}$ is the probability that the state at time n is j given that the initial state at time 0 is i.

$$P_{ij}^{(n)} = P_r[X_n = j \mid X_0 = i]$$

If the unconditional distribution of the state at time n is desired, it is necessary to specify the probability distribution of the initial state. Let us denote this by

$$\alpha_i = P_r\{X_0 = i\} \quad ; \quad i \geq 0 \left(\sum_{i=0}^{\infty} \alpha_i = 1 \right)$$

So,

$$P_r\{X_n = j\} = \sum_{i=0}^{\infty} P_r[X_n = j, X_0 = i]$$

$$= \sum_{i=0}^{\infty} P_r[X_n = j \mid X_0 = i] \, P_r[X_0 = i]$$

$$= \sum_{i=0}^{\infty} P_{ij}^{(n)} \alpha_i$$

Example:

Consider a Markov chain with the following TPM

$$P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 2/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

with the initial distribution $\alpha_i = P[X_0 = i] = 1/3 \; ; i = 0,1,2$

Find

1. $P_{01}^{(2)}, \; P_{02}^{(2)}$ 2. $P[X_2 = 1, X_0 = 0]$

3. $P(0,1,1) = P(X_0 = 0, X_1 = 1, X_2 = 1)$ 4. $P[X_1 = 2]$

5. $P(0,0,1,1) = P(X_0 = 0, X_1 = 0, X_2 = 1, X_3 = 1)$

Solution:

(1) $P^2 = P \cdot P$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 2/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array} \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 2/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

$P_{01}^{(2)}$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 5/8 & 5/16 & 1/16 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{bmatrix} \end{array}$$

$\therefore P_{01}^{(2)} = P[X_2 = 1 \mid X_0 = 0]$

$= 5/16$

So, the Markov chain is in state 0 to state 1 at ~~2nd~~ 2nd step with probability 5/16

$$P_{02}^{(2)} = P[X_2 = 2 \mid X_0 = 0] = 1/16$$

$\therefore$ ~~The~~ $P_{02}^{(2)}$ implies that the probability of transission of markov chain from state 0 to 2 at 2nd step with probability 1/16.

Solution (2):

We know,

$$P[X_2 = 1, X_0 = 0] = P[X_2 = 1 \mid X_0 = 0] \times P[X_0 = 0]$$

$$= \frac{5}{16} \times \frac{1}{3}$$

$$= \frac{5}{48}$$

Solution (3):

We know,

$$P(0,1,1) = P(X_0=0,\ X_1=1,\ X_2=1)$$

$$= P(X_2=1 \mid X_1=1,\ X_0=0) \times P(X_1=1,\ X_0=0)$$

$$= P(X_2=1 \mid X_1=1) \times P(X_1=1 \mid X_0=0) \times P(X_0=0)$$

$$= \frac{2}{4} \times \frac{1}{4} \times \frac{1}{3}$$

$$= \frac{1}{24}$$

∴ The joint probability of Markov chain in state 0, 1 and 1 is $\frac{1}{24}$

Solution (4): We know that,

$$P_r\{X_n=j\} = \sum_{j=0}^{\infty} P_{ij}^{(n)} \alpha_i \text{. Thus}$$

$$P[X_1=2] = \sum_{j=0}^{2} P_{i2}^{(1)} \alpha_i \qquad i=0,1,2 \quad j=0,1,2$$

$$= P_{02}\alpha_0 + P_{12}\alpha_1 + P_{22}\alpha_2$$

$$= 0 \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3}$$

$$= \frac{1}{6}$$

∴ Irrespective of initial state the Markov cha

CamScanner

will be in state 2 at 1st step is 1/6

Solution (5):

We know that

$$P(0,0,1,1) = P(X_0=0,\ X_1=0,\ X_2=1,\ X_3=1)$$

$$= P[X_3=1 \mid X_2=1]\, P[X_2=1 \mid X_1=0] \times P[X_1=0 \mid X_0=0]\, P[X_0=0]$$

$$= \frac{2}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3}$$

$$= \frac{1}{32}$$

Also find: $P(0,1,2)$ and ~~P(0~~ $P(1,0,2)$

① $P(0,1,2)$

$$= P_r(X_0=0,\ X_1=1,\ X_2=2)$$

$$= P_r(X_2=2 \mid X_1=1)\, P_r(X_1=1 \mid X_0=0)\, P_r(X_0=0)$$

$$= P_{12} \times P_{01} \times P_r(X_0=0)$$

$$= \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{3}$$

$$= \frac{1}{48}$$

Q2 $P(1,0,2)$

$= P_r(X_0 = 1,\ X_1 = 0,\ X_2 = 2)$

$= P_r(X_2 = 2 \mid X_1 = 0)\ P_r(X_1 = 0 \mid X_0 = 1)\ P_r(X_0 = 1)$

$= P_{02} \cdot P_{10} \cdot P(X_0 = 1)$

$= 0 \cdot \frac{1}{4} \cdot \frac{1}{3}$

$= 0$

Chapman-Kolmogorov Equation:

Chapman-Kolmogorov equation provides a method for finding higher-step transition probabilities. If $\{X_n\}_{n=0}^{\infty}$ is a Markov chain then for ~~a~~ $m, n > 0$

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)}\, P_{kj}^{(n)}$$

Proof: $\cancel{P_{ij}^{(m+n)} = P_r\{X_{m+n} = j,\ X_m = k \mid X_0 = i\}}$

$\cancel{= \sum_{k=0}^{\infty} P_r\{X_{m+n}}$

$$P_{ij}^{(m+n)} = P_r\{X_{m+n} = j \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_r\{X_{m+n} = j,\ X_m = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_r\{X_{m+n} = j \mid X_m = k, X_0 = i\}\ P_r\{X_m = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P_r \{X_{m+n} = j \mid X_m = k\} \, P_r \{X_m = k \mid X_0 = i\}$$

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{kj}^{(n)} \, P_{ik}^{(m)}$$

$$\therefore P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} \, P_{kj}^{(n)}$$

(Proved)

Theorem:

Prove that, if state i communicate with state j and state j communicate with state k then state i communicate with state k i.e., $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

Proof:

Since i and j communicate with each other that is, $i \leftrightarrow j$

$\therefore P_{ij}^{(m)} > 0$ and $P_{ji}^{(s)} > 0$

and $j \leftrightarrow k$ implies that, $P_{jk}^{(n)} > 0$ and $P_{kj}^{(t)} > 0$

Now, according to Chapman Kolmogorov equation:

$$P_{ik}^{(m+n)} = \sum_{l=0}^{\infty} P_{il}^{(m)} P_{lk}^{(n)} = \sum_{l=j} P_{il}^{(m)} P_{lk}^{(n)} + P_{ij}^{(m)} P_{jk}^{(n)}$$

$$\therefore P_{ik}^{(m+n)} \geq P_{ij}^{(m)} P_{jk}^{(n)}$$

$$\therefore P_{ik}^{(m+n)} > 0 \qquad [\because P_{ij}^{(m)} > 0 \text{ and } P_{jk}^{(n)} > 0]$$

Thus, $i \to k$. i.e. k is accessible from state i

Again,

$$P_{ki}^{(s+t)} = \sum_{q=0}^{\infty} P_{kq}^{(t)} P_{qi}^{(s)}$$

$$= \sum_{q \neq j} P_{kq}^{(t)} P_{qi}^{(s)} + P_{kj}^{(t)} P_{ji}^{(s)}$$

$$\therefore P_{ki}^{(s+t)} \geq P_{kj}^{(t)} P_{ji}^{(s)}$$

i.e., $P_{ki}^{(s+t)} > 0 \qquad [\because P_{kj}^{(t)} > 0 \text{ and } P_{ji}^{(s)} > 0]$

Thus, $k \to i$. i.e. state i is accessible from state k

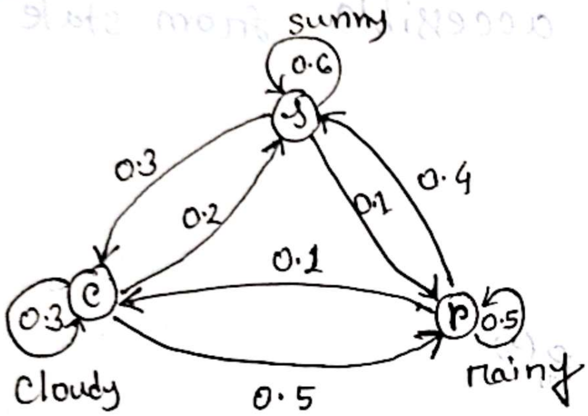
Since, $P_{ik}^{(m+n)} > 0$ and $P_{ki}^{(s+t)} > 0$, thus, ∃ so that $i \leftrightarrow k$

This implies that if state i communicate with state j and state j communicate with state k then state i also communicate with state k.

Transition graph

The graphical representation of the Transition probability Matrix (TPM) of a Markov chain is called transition graph.

Example: State space $S = \{s, c, r\}$



$$P = \begin{array}{c c} & \begin{array}{ccc} s & c & r \end{array} \\ \begin{array}{c} s \\ c \\ r \end{array} & \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \end{array}$$

L - 7 & 8

Example 1:

Solve (i): Transition matrix

$$TM = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 4 & 2 & 0 \\ 3 & 9 & 2 \\ 0 & 2 & 2 \end{bmatrix} \end{array}$$

$\therefore$ Transition probability matrix

$$TPM = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 3/14 & 9/14 & 1/7 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Solution (ii)

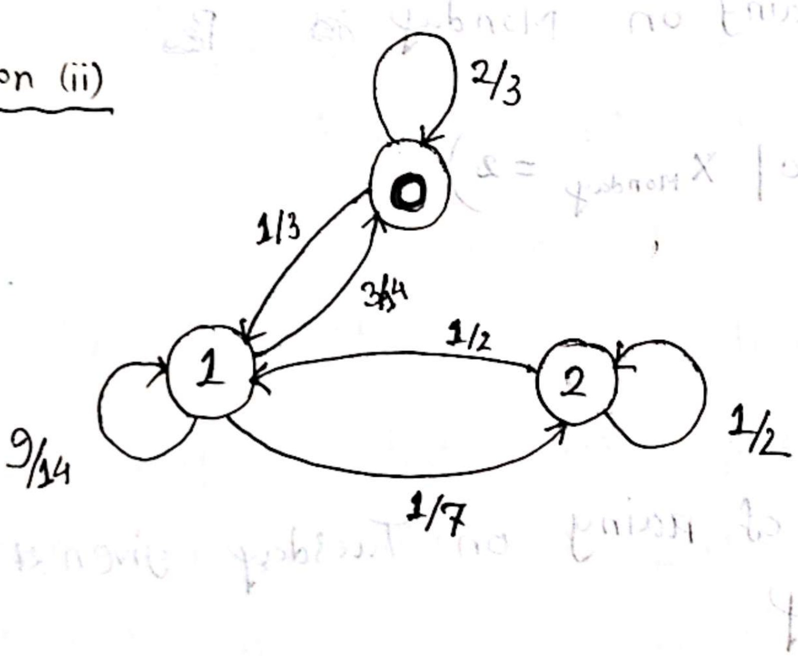


Fig: Transition probability diagram.

Solution (iii)

Let X_n represents the daily weather, ~~for~~ where

$$X_n = \begin{cases} 0; \text{ sunny} \\ 1; \text{ cloudy} \\ 2; \text{ Rainy} \end{cases}$$

From (i) we have,

$$P = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 3/14 & 9/14 & 1/7 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Probability of sunny weather on Tuesday given that it was rainy on Monday is P_{20}.

$$\therefore P(X_{\text{Tuesday}} = 0 \mid X_{\text{Monday}} = 2)$$

$$= P_{20}$$

$$= 0$$

(iv) Probability of rainy on Tuesday given cloudy on saturday

$P(\cancel{X_{\text{saturday}} = 2 \mid X}$

$$P(X_{\text{Tuesday}} = 2 \mid X_{\text{saturday}} = 1) = P_{12}^{(3)}$$

CamScanner

~~Now~~, Now

$$P^3 = P \cdot P^2 \cdot P$$

$$= \begin{bmatrix} 2/3 & 1/3 & 0 \\ 3/14 & 9/14 & 1/7 \\ 0 & 1/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 2/3 & 1/3 & 0 \\ 3/14 & 9/14 & 1/7 \\ 0 & 1/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 2/3 & 1/3 & 0 \\ 3/14 & 9/14 & 1/7 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 & 0 \\ 3/14 & 9/14 & 1/7 \\ 0 & 1/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 0.515 & 0.436 & 0.047 \\ 0.281 & 0.556 & 0.163 \\ 0.107 & 0.572 & 0.322 \end{bmatrix}$$

$$= \begin{bmatrix} 0.437 & 0.476 & 0.086 \\ 0.306 & 0.533 & 0.161 \\ 0.194 & 0.566 & 0.243 \end{bmatrix}$$

$\therefore \quad P_{12}^{(3)} = 0.161$

This implies that the probability of ~~sunny~~ rainy weather on tuesday given that it was cloudy ~~rainy~~ on ~~Monday~~ Saturday is 0.161. Therefore 16.1% chance that weather will be rainy on tuesday after three days.

(v) The classes of Markov chain is:

$\{0, 1, 2\}$

$$F_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)}$$

$$= f_{00}^{(1)} + f_{00}^{(2)} + \cdots$$

$$= \frac{2}{3} + \cancel{0.515 + \cdots}$$

$$= \frac{2}{3} + \cdots$$

$$= 0.667 + \cdots$$

$\therefore F_{00} \cong 1$

So, state '0' is recurrent

Similarly state "1" and state "2" is also recurrent.

(vi) From (i), class of states : $\{0, 1, 2\}$

Here, the Markov chain does't contain any other proper closed subset of states other than the state space, So this Markov chain is irreducible.

Since $P_{00} > 0$, $P_{11} > 0$, $P_{22} > 0$

Thus $P_{ii}^{(1)} > 0$

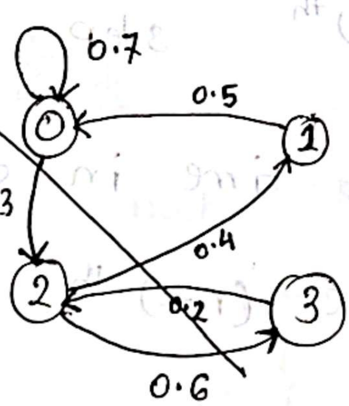
$\therefore$ All states are aperiodic

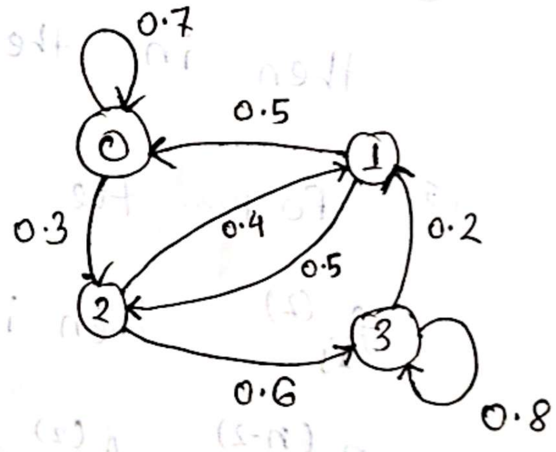
Class work :

Solution (i):

$$\text{TPM} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \end{array}$$

~~Solution (ii)~~





Solution (ii) :

Fig: Transition Probability Graph.

L-08, 10

~~Theorem: Show that, for any state J, $\sum^{\infty}$~~

First entrance decomposition formula.

for Markov chain $\{X_n\}_{n=0}^{\infty}$

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{jj}^{(n-r)} \; ; \quad n \geq r$$

Proof:

Starting from state i, the process will enter state j is the union of the following mutually exclusive events.

① For the first time in first step f_{ij}, then in the $(n-1)^{th}$ step $P_{jj}^{(n-1)} \cdot f_{jj}^{(1)} P_{jj}^{(n-1)}$

② For the first time in second step $f_{ij}^{(2)}$, then in the $(n-2)^{th}$ step

$P_{jj}^{(n-2)} \cdot f_{jj}^{(2)} P_{jj}^{(n-2)}$

③ for the first time in r^{th} $f_{ij}^{(r)}$ then in the $(n-r)^{th}$ step $P_{jj}^{(n-r)} \cdot f_{ij}^{(r)} P_{jj}^{(n-r)}$

CamScanner

$\therefore P_{ij}^{(n)} = f_{ij}^{(1)} P_{jj}^{(n-1)} + f_{jj}^{(2)} P_{jj}^{(n-2)} + \cdots + f_{jj}^{(r)} P_{jj}^{(n-r)}$

$$= \sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)}$$

(Proved)

L-9 & 10

Theorem: Show that, for any state j, $\sum_{n=0}^{\infty} P_{jj}^{(n)} = \frac{1}{1-F_{jj}}$ and if state j is recurrent then $\sum_{n=0}^{\infty} P_{jj}^{(n)} = \infty$

Proof: We know,

$$P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)}$$

Multipling both sides by t^n and adding for all $n \geq 1$, we get

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} t^n = \sum_{r=0}^{n} \sum_{n=1}^{\infty} f_{jj}^{(r)} P_{jj}^{(n-r)} t^n$$

$$= \sum_{r=0}^{n} f_{jj}^{(r)} t^r \sum_{n=1}^{\infty} P_{jj}^{(n-r)} t^{n-r}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_{jj}^{(n)} t^n - 1 = \sum_{n=0}^{n} f_{jj}^{(n)} t^n \cdot \sum_{n=0}^{\infty} P_{jj}^{(n)} t^n \quad \text{—(i)}$$

Put $t=1$ in equation (i), we get

$$\sum_{n=0}^{\infty} P_{jj}^{(n)} - 1 = \sum_{n=0}^{n} f_{jj}^{(n)} \sum_{n=0}^{\infty} P_{jj}^{(n)}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_{jj}^{(n)} \left[1 - \sum_{n=0}^{n} f_{jj}^{(n)}\right] = 1$$

$$\therefore \sum_{n=0}^{\infty} P_{jj}^{(n)} = \frac{1}{1-F_{jj}}$$

(Proved)

Theorem: From state j the mean number of returns to state j is $\frac{F_{jj}}{1-F_{jj}}$

Proof: Let,

$$R_n = \begin{cases} 1; & X_n = j \\ 0; & X_n \neq j \end{cases}$$

$\therefore \sum_{n=1}^{\infty} R_n$ is the total number of return to state j

$$\therefore E\left[\sum_{n=1}^{\infty} R_n \mid X_0 = j\right] = \sum_{n=1}^{\infty} E\left[R_n \mid X_0 = j\right]$$

CamScanner

$$= \sum_{n=1}^{\infty} \left[1 \cdot P_r\{X_n = j \mid X_0 = j\} + 0 \cdot P_r\{X_n \neq j \mid X_0 = j\} \right]$$

$$\therefore E\left[\sum_{n=1}^{\infty} R_n\right] = \sum_{n=1}^{\infty} P_r\{X_n = j \mid X_0 = j\}$$

$$= \sum_{n=1}^{\infty} P_{jj}^{(n)}$$

$$= \sum_{n=0}^{\infty} P_{jj}^{(n)} - 1$$

$$= \frac{1}{1 - F_{jj}} - 1$$

$$\therefore E\left[\sum_{n=1}^{\infty} R_n\right] = \frac{F_{jj}}{1 - F_{jj}}$$

(Proved)

<u>Theorem:</u> If state i is recurrent and $i \leftrightarrow j$ then state j is also recurrent

<u>Proof:</u> Since $i \leftrightarrow j$

$$P_{ij}^{(k)} > 0 \quad \text{and} \quad P_{ji}^{(m)} > 0$$

Since i in recurrent

$$\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$$

$$P_{jj}^{(m+n+k)} = \sum_{i=0}^{\infty} P_{ji}^{(m)} \quad P_{ij}^{(n+k)}$$

$$\therefore \; P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} \; P_{ij}^{(n+k)} \quad \text{——— (1)}$$

Again,

$$P_{ij}^{(n+k)} = \sum_{i=0}^{\infty} P_{ii}^{(n)} \; P_{ij}^{(k)}$$

$$\therefore \; P_{ij}^{(n+k)} \geq P_{ii}^{(n)} \; P_{ij}^{(k)} \quad \text{——— (2)}$$

combining equation (1) & (2), we get

$$P_{jj}^{(m+n+k)} \quad P_{ij}^{(n+k)} \geq P_{ji}^{(m)} . P_{ij}^{(n+k)} . \quad P_{ii}^{(n)} . P_{ij}^{(k)}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} . \; P_{ij}^{(k)} \sum_{n=0}^{\infty} P_{ii}^{(n)}$$

$$\therefore \; \sum_{n=0}^{\infty} P_{jj}^{(m+n+k)} = \infty$$

Hence, $\displaystyle\sum_{n=0}^{\infty} P_{jj}^{(n)} = \infty$

(Proved)

Example 1: Given transition probability matrix

$$P = \begin{bmatrix} 0.1 & 0.6 & 0 & 0.3 \\ 0.3 & 0 & 0.5 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$P(R) = 1$

$P(C) = 0$

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} P_{jj}^{(n-r)}$$

$$P(M) = 0.1 \cdot P(M) + 0.6\, P(S) + 0.3\, P(R)$$

$$\Rightarrow 0.9\, P(M) = 0.6\, P(S) + 0.3$$

$$\therefore P(M) = \frac{0.6\, P(S) + 0.3}{0.9} \quad \text{—— (1)}$$

$$\Rightarrow P(S) = 0.3\, P(M) + \cancel{0.2}\ 0.5\, P(C) + 0.2\, P(R)$$

$$\Rightarrow P(S) = 0.3\, P(M) + 0.2 \quad \text{—— (2)}$$

Put the value ~~of~~ (2) in equation (1).

we get,

$$P(M) = \frac{0.6\{0.3\, P(M) + 0.2\} + 0.3}{0.9}$$

$$\Rightarrow P(M) = 0.2\, P(M) + 0.467$$

CamScanner

$\Rightarrow 0.8 P(M) = 0.467$

$\therefore P(M) = 0.583$

$\therefore$ The patient has a 58.3% chance of recovering before entering a critical condition.

<u>Example 2:</u> Here, X_n represents the states of hypothetical stock market.

$$\therefore X_n = \begin{cases} 1 ; & \text{bull} \\ 2 ; & \text{bear} \\ 3 ; & \text{Stagnent} \end{cases}$$

The transition probability matrix is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

(i) Class $\{1, 2, 3\}$

For state '1',

$P_{11}^{(1)} = 0.9 \quad , \quad P_{11}^{(2)} = 0.83, \ldots$

$\therefore F_{11}^{\epsilon} = P_{11}^{(1)} + \ldots$

$= 0.90 \ldots$

CS CamScanner

$\therefore F_{11} \cong 1$ $\therefore$ State '1' is recurrent

Since, state '1', '2' and '3' communicate with each other, state '2' and '3' are also recurrent.

(ii) From (i), class of states : $\{1, 2, 3\}$

Here, the Markov chain does not contain any other proper closed subset other than the state space, so this Markov chain is irreducible

Since, $P_{11}^{(1)} > 0$, therefore, the periodicity of state '1' is 1. So, state '1' is aperiodic.

Similarly state '2' and '3' are also aperiodic.

(iii) From (i)

$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

Now,

$$\cancel{\pi_1 = 0.9\pi_1 + 0.075\pi_2 + 0.025\pi_3}$$

$$\cancel{\pi_2 = 0.15}$$

Now,

$$\pi_1 = 0.9\pi_1 + 0.15\pi_2 + 0.25\pi_3 \quad \text{——} (1)$$

$$\pi_2 = 0.075\pi_1 + 0.8\pi_2 + 0.25\pi_3 \quad \text{——} (2)$$

$$\pi_3 = 0.025\pi_1 + 0.05\pi_2 + 0.5\pi_3 \quad \text{——} (3)$$

and,

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\therefore \pi_1 = 1 - \pi_2 - \pi_3 \quad \text{——} (4)$$

From (1) $\Rightarrow$

$$\therefore 1 - \pi_2 - \pi_3 = 0.9(1 - \pi_2 - \pi_3) + 0.15\pi_2 + 0.25\pi_3$$

$$\Rightarrow 1 - \pi_2 - \pi_3 = 0.9 - 0.9\pi_2 - 0.9\pi_3 + 0.15\pi_2 + 0.25\pi_3$$

$$\Rightarrow 1 - 0.9 = \pi_2 + \pi_3 - 0.9\pi_2 - 0.9\pi_3 + 0.15\pi_2 + 0.25\pi_3$$

$$\Rightarrow 0.1 = 0.25\pi_2 + 0.35\pi_3$$

$$\therefore 0.25\pi_2 + 0.35\pi_3 = 0.1 \quad \text{——} (5)$$

From (2)

$$\pi_2 = 0.075(1-\pi_2-\pi_3) + 0.8\pi_2 + 0.25\pi_3$$

$$\Rightarrow \pi_2 + 0.075\pi_2 - 0.8\pi_2 + 0.075\pi_3 - 0.25\pi_3 = 0.075$$

$$\therefore 0.275\pi_2 - 0.175\pi_3 = 0.075 \quad \text{—— (6)}$$

By solving eq^{n} (5) & (6), we get

$$\pi_2 = \frac{5}{16}$$

$$\pi_3 = \frac{1}{16}$$

From (4),

$$\pi_1 = 1 - \frac{5}{16} - \frac{1}{16} = \frac{5}{8}$$

$\therefore$ The long term fraction of weeks during which, the market is

Stagnant $= \pi_3 \times 100\% = 6.25\%$

bull $= \pi_1 \times 100\% = \frac{5}{8} \times 100\% = 62.5\%$

bear $= \pi_2 \times 100\% = \frac{5}{16} \times 100\% = 31.25\%$

Class Work:

Example-3: Given that,

$$P = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}$$

Let X_n represents the states of social classes

$$\therefore X_n = \begin{cases} 0; & \text{upper-class} \\ 1; & \text{middle-class} \\ 2; & \text{lower-class} \end{cases}$$

(i) The limiting probabilities π_i thus satisfy:

$$\pi_0 = 0.45\pi_0 + 0.05\pi_1 + 0.01\pi_2 \quad \text{---} (1)$$

$$\pi_1 = 0.48\pi_0 + 0.70\pi_1 + 0.50\pi_2 \quad \text{---} (2)$$

$$\pi_2 = 0.07\pi_0 + 0.25\pi_1 + 0.49\pi_2 \quad \text{---} (3)$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\therefore \pi_0 = 1 - \pi_1 - \pi_2 \quad \text{---} (4)$$

CamScanner

From (1)

$$1-\pi_1-\pi_2 = 0.45(1-\pi_1-\pi_2)+0.05\pi_1+0.01\pi_2$$

$$\Rightarrow 1-\pi_1-\pi_2 = 0.45-0.45\pi_1-0.45\pi_2+0.05\pi_1+0.01\pi_2$$

$$\Rightarrow 1-0.45 = \pi_1+\pi_2+0.45\pi_1+0.45\pi_2+0.05\pi_1+$$
$$0.01\pi_2$$

$$\therefore 0.6\pi_1+0.56\pi_2=0.55 \quad \text{—— (5)}$$

$$\pi_1 = 0.48(1-\pi_1-\pi_2)+0.70\pi_1+0.50\pi_2$$

$$\Rightarrow \pi_1 = 0.48-0.48\pi_1-0.48\pi_2+0.70\pi_1+0.50\pi_2$$

$$\Rightarrow \pi_1+0.48\pi_1+0.48\pi_2-0.70\pi_1-0.50\pi_2 = 0.48$$

$$\therefore 0.78\pi_1-0.02\pi_2=0.48 \quad \text{—— (6)}$$

By solving equation (5) & (6), we get

$\pi_1 = 0.62$

$\pi_2 = 0.31$

$\therefore$ From (4)

$$\pi_0 = 1-0.62-0.31 = 0.07$$

CamScanner

$\therefore$ In the long run 7% of people in upper class jobs, 62% people in middle class job and 31% people in lower class jobs.

CamScanner

Counting Process and Poission Process

Derivation of Poission Process:

Statement:

If $[N(t), t \geq 0]$ be a Poission process with parameter λ, then under certain assumptions $N(t)$ follows Poission distribution with mean λt, i.e.,

$$P_n(t) = Pr\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad ; \; n = 0, 1, 2, \ldots$$

Proof:

Assumptions:

(1) Number of occurance in disjoint time interval are independent.

(2) The distribution of number of occurance depends only on the length of time not on the end point.

(3) The number of occurance at 0 time, $N(0) = 0$

(4) $P_0(0) = 1$

(5) $P[N(h)=1] = \lambda h + o(h)$

(6) $[N(h) \geq 2] = o(h)$

In order to derive the distribution,

$$P_0(t+h) = P_r\{N(t+h)=0\}$$

$$= P_r\{N(t)=0,\ N(t+h)-N(t)=0\}$$

$$= P_r\{N(t)=0\} \cdot P_r\{[N(t+h)-N(t)]=0\}$$

$$= P_r\{N(t)=0\} \cdot P_r\{N(h)=0\}$$

$$= P_0(t)\left[1 - P_r\{N(h) \geq 1\}\right]$$

$$= P_0(t)\left[1 - \lambda h + o(h)\right]$$

$$= P_0(t) - \lambda h\, P_0(t) + o(h) \cdot P_0(t)$$

$$\Rightarrow P_0(t+h) - P_0(t) = -\lambda h\, P_0(t) + o(h) \cdot P_0(t)$$

$$\Rightarrow \lim_{h \to 0} \frac{P_0(t+h) - P_0(t)}{h} = \cancel{-\lambda h} - \lambda P_0(t) + \lim_{h \to 0} \frac{o(h)}{h} \cdot P_0(t)$$

$$\Rightarrow P_0'(t) = -\lambda P_0(t)$$

$$\Rightarrow \frac{P_0'(t)}{P_0(t)} = -\lambda$$

$$\Rightarrow \int \frac{P_0'(t)}{P_0(t)}\,dt = -\int \lambda\,dt$$

$$\Rightarrow \ln P_0(t) = -\lambda t + c$$

$$\Rightarrow P_0(t) = e^{-\lambda t + c}$$

$$\Rightarrow P_0(t) = k e^{-\lambda t} \quad [\because k = e^c] \qquad \text{---(i)}$$

Putting $t = 0$

$P_0(0) = k$

$\therefore k = 1$

$$\therefore P_0(t) = e^{-\lambda t} \qquad \text{---(ii)}$$

$$P_n(t+h) = Pr\{N(t+h) = n\}$$

$$= Pr\{N(t) = n, N(h) = 0\} + Pr\{N(t) = n-1, N(h) = 1\}$$
$$+ Pr\{N(t) = n-2, N(h) = 2\} + \cdots$$

$$= P_n(t)\,Pr\{N(h) = 0\} + P_{n-1}(t)\,Pr\{N(h) = 1\}$$
$$+ P_{n-2}(t) \cdot Pr\{N(h) = 2\} + \cdots$$

$$= P_n(t)[1 - \lambda h + o(h)] + P_{n-1}(t)[\lambda h + o(h)]$$
$$+ P_{n-2}(t)[o(h)] + o(h)$$

CamScanner

$$= P_n(t) - \lambda h P_n(t) + P_n(t)\,o(h) + \lambda h P_{n-1}(t) + P_{n-1}(t)\,o(h) + P_{n-2}(t)\,o(h) + o(h)$$

$$\Rightarrow P_n(t+h) - P_n(t) = -\lambda h P_n(t) + \lambda h P_{n-1}(t) + P_n(t)\,o(h) + P_{n-1}(t)\,o(h) + P_{n-2}(t)\,o(h) + o(h)$$

$$\Rightarrow \lim_{h\to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + P_n(t)\lim_{h\to 0}\frac{o(h)}{h} + P_{n-1}(t) \times \lim_{h\to 0}\frac{o(h)}{h} + P_{n-2}(t)\lim_{h\to 0}\frac{o(h)}{h} + \lim_{h\to 0}\frac{o(h)}{h}$$

$$\Rightarrow P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) + 0 + 0 + 0 + 0$$

$$\Rightarrow P_n'(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

$$\Rightarrow e^{\lambda t}\left[P_n'(t) + \lambda P_n(t)\right] = \lambda e^{\lambda t} P_{n-1}(t) \quad \text{———} \cancel{(iii)}$$

Putting $n = 1$ ~~in equation (iii), we get~~

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_n(t)\right] = \lambda e^{\lambda t} P_{n-1}(t) \quad \text{———} (iii)$$

CamScanner

Putting $n=1$ in equation (iii), we get

$$\frac{d}{dt}\left[e^{\lambda t} P_1(t)\right] = \lambda e^{\lambda t} \cdot e^{-\lambda t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_1(t)\right] = \lambda$$

$$\Rightarrow e^{\lambda t} P_1(t) = \lambda t + c$$

If $t=0$

$c = 0$

$$\therefore e^{\lambda t} P_1(t) = \lambda t$$

$$\therefore P_1(t) = \lambda t e^{-\lambda t} \quad \text{——— (iv)}$$

Putting $n=2$ in equation (ii), we get

$$\frac{d}{dt}\left[e^{\lambda t} P_2(t)\right] = \lambda e^{\lambda t} \cdot \lambda t e^{-\lambda t}$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda t} P_2(t)\right] = \lambda^2 t$$

$$\Rightarrow e^{\lambda t} P_2(t) = \frac{\lambda^2 t^2}{2} + c$$

If $t=0$

$\therefore c=0$

$$\therefore e^{\lambda t} P_2(t) = \frac{(\lambda t)^2}{2!}$$

So, $P_2(t) = \dfrac{(\lambda t)^2 e^{-\lambda t}}{2!}$

$$P_3(t) = \frac{e^{-\lambda t} (\lambda t)^3}{3!}$$

$\vdots$

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

Properties - 3: ~~Sum~~

Sum of independent poisson is also a Poisson process.

Proof:

Probability generating function

$P.g.f = e^{\lambda(s-1)t}$ ~~[i=1,2, ..., N]~~

Suppose $N_i(t)$ $[i=1,2,\dots,n]$ be independent ~~be~~ Poisson process with rate λ_i $[i=1,2,\dots,n]$ respectively.

The $P.g.f$ of $\sum N_i(t) = \sum_{i=1}^{n} e^{\lambda_i(s-1)t}$

$$= e^{\lambda_1(s-1)t} \cdot e^{\lambda_2(s-1)t} \, e^{\lambda_3(s-1)t} \dots$$

$$= e^{\lambda_1(s-1)t} \; e^{\lambda_2\left(\frac{1}{s}-1\right)t}$$

$$= e^{\lambda_1 ts} \cdot e^{-\lambda_1 t} \cdot e^{\frac{\lambda_2}{s}t} \cdot e^{-\lambda_2 t}$$

$$\cancel{= e^{-(\lambda_1+\lambda_2}}$$

$$\cancel{= e^{\lambda_1 ts} \; e^{-\lambda_1 t}}$$

$$\therefore E[s^{N(t)}] = e^{-(\lambda_1+\lambda_2)t} \; e^{(\lambda_1 ts + \lambda_2 t/s)}$$

which is not a pgf of a Poisson process.

Property-6:

If $\{M(t), t \geq 0\}$ is Poisson Process and $s < t$ then $Pr\{N(s)=k \,|\, N(t)=n\} = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$

Proof: Given,

$$Pr\{N(s)=k \,/\, N(t)=n\} = \frac{Pr\{N(s)=k \text{ and } N(t)=n\}}{Pr\{N(t)=n\}}$$

$$= \frac{Pr\{N(s)=k \text{ and } N(t-s)=n-k\}}{Pr\{N(t)=n\}}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{k!} \times \dfrac{e^{-\lambda(t-s)}\{\lambda(t-s)\}^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!} \; \frac{e^{-\lambda s}(\lambda s)^k \; e^{\lambda s}\{\lambda(t-s)\}^{n-k}}{(\lambda t)^n}$$

$$= \frac{n!}{k!\,(n-k)!}\left(\frac{\lambda s}{\lambda t}\right)^k \left\{\frac{\lambda(t-s)}{\lambda t}\right\}^{n-k}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$$

(Proved)

Property-7:

Suppose $[N(t), t \geq 0]$ be a Poisson Process then the Auto correlation Coefficient between $N(t)$ and $N(t+$

Property-7:

Suppose $[N(t), t \geq 0]$ be a Poisson Process then the Autocorrelation coefficient between $N(t)$ and $N(t+s)$ is $\sqrt{\dfrac{t}{t+s}}$

Proof:

Let, λ be the parameter of the process then,

$$E[N(t)] = \lambda t, \quad var[N(t)] = \lambda t$$

$$\cancel{E[N_2} \; E[N^2(t)] = \lambda t + (\lambda t)^2 \qquad \left[\because V[N(t)] = E[N^2(t)] - E\{N(t)\}^2\right]$$

$$E[N(t) \cdot N(t+s)] = E[N(t)][N(t+s) - N(t) + N(t)]$$

$$= E[\{N(t)\}^2] + E\left[N(t)\left[N(t+s) - N(t)\right]\right]$$

$$= E[N(t)^2] + E[N(t) \cdot N(s)]$$

$$= \lambda t + (\lambda t)^2 + E[N(t)] \cdot E[N(s)]$$

$$= \lambda t + (\lambda t)^2 + \lambda t \cdot \lambda s$$

$$= \lambda t + (\lambda t)^2 + \lambda^2 ts$$

CamScanner

$\therefore \operatorname{cov}[N(t)\cdot N(t+s)] = E[N(t)\cdot N(t+s)] - E[N(t)]\cdot E[N(t+s)]$

$$= \lambda t + (\lambda t)^2 + \lambda^2 ts - [\lambda t \cdot \lambda(t+s)]$$

$$= \lambda t + (\lambda t)^2 + \lambda^2 ts - \lambda^2 t^2 - \lambda^2 ts$$

$$= \lambda t$$

$\therefore$ Correlation $\{N(t)\cdot N(t+s)\} = \dfrac{\operatorname{cov}\{N(t)\cdot N(t+s)\}}{\sqrt{\operatorname{Var}\{N(t)\}\cdot \operatorname{Var}\{N(t+s)\}}}$

$$= \frac{\lambda t}{\sqrt{\lambda t \cdot \lambda(t+s)}}$$

$$= \frac{\lambda t}{\sqrt{\lambda^2 t(t+s)}}$$

$$= \sqrt{\frac{t^2}{t(t+s)}}$$

$\therefore$ correlation $= \sqrt{\dfrac{t}{t+s}}$

(Proved)

Example 1:

Suppose that customers arrive at a Bank according to a Poisson process with mean rate λ per minute. Then the number of customers $N(t)$ arriving in an interval of duration t minutes follows Poisson distribution with mean λt. If the rate of arrival is 3 per minutes, then in an arrival of 2 minutes, find the probability that the number of customer arriving is:

Solution:

$N(t)$ follows Poisson distribution with mean λt.

$$\therefore \Pr\{N(t)=n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Here, $t=2$, $\lambda=3$

(i) $$\Pr\{N(2)=4\} = \frac{e^{-(3\times2)}(3\times2)^4}{4}$$

$$= 0.134$$

(ii) $\Pr\{N(2) > 4\} = 1 - \Pr\{N(2) \le 4\}$

$= 1 - [\Pr\{N(2) = 0\} + \Pr\{N(2) = 1\} + \Pr\{N(2) = 2\} + \Pr\{N(2) = 3\} + \Pr\{N(2) = 4\}$

$= 1 - [0.002478 + 0.01487 + 0.044617 + 0.089235 + 0.13385]$

$\therefore \Pr\{N(2) > 4\} = 0.715$

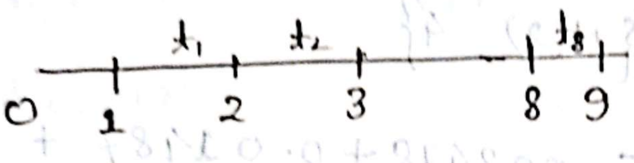
(iii) $\Pr\{N(2) < 4\} = \Pr\{N(2) = 0\} + \Pr\{N(2) = 1\} + \Pr\{N(2) = 2\} + \Pr\{N(2) = 3\}$

$= 0.1512$

Interarrival Time:

Proof:

Let X be the inter arrival time between successive occurrance



Let two successive occurrance E_i and E_{i+1}. suppose ~~where~~ E_i ~~at t_i~~ occured at the instant t_i. Then

$Pr\{X \le x\} = F(x) =$ distribution function

$Pr\{X > x\} = Pr\{E_{i+1}$ did not occur within time $(t_i, t_{i+x}) \mid E_i$ occured at time $t_i\}$

$= Pr[\{E_{i+1}$ did not occur in $(t_i, t_{i+x})\} \mid N(t_i) = 1]$

$$= Pr\{N(x) = 0 \mid N(t_i) = 1\}$$

$$= Pr\{N(x) = 0\}$$

$$= \frac{e^{-\lambda x}(\lambda x)^0}{0!}$$

CS CamScanner

$$= e^{-\lambda x}$$

$$Pr\{X \leq x\} = 1 - Pr\{X > x\}$$

$$\therefore F(x) = 1 - e^{-\lambda x}$$

$$\therefore Pr\{X\} = \frac{d}{dx} F(x) = \lambda e^{-\lambda x} \quad ; x > 0$$

Example-2:

Suppose car passes Prantic Gate at a Poisson rate of 1 per minute. If 5% of the cars are Toyota then

a) What is the probability that at least one Toyota passes by during an hour.

b) Given that 10 Toyotas had passed by an hour, what is the expected number of cars has passed by at ~~time~~ that time

c) If 50 cars have passed by an hour, what is the probability that 5 of them are Toyotas.

CamScanner

Solution:

a) Suppose $N(t)$ = Total car

$N_1(t)$ = Number of Toyota

$N_2(t)$ = Number of others

$N(t) = \lambda = 1$ min

Here,

$P = 5\% = \frac{5}{100} = 0.05$

$q = 0.95$

$\lambda_1 = N_1(t) = \lambda P = 1 \times 0.05 = 0.05$

$\lambda_2 = N_2(t) = \lambda q = 1 \times 0.95 = 0.95$

$$P_r\{N_1(60) \geq 1\} = 1 - P_r\{N_1(60)\}$$

$$= 1 - P_r\{N_1(60) = 0\}$$

$$= 1 - \frac{e^{-0.05\times60} \times (0.05\times60)^0}{0!}$$

$$\therefore P_r\{N_1(60) \geq 1\} = 0.9502$$

CamScanner

b) $E[N(60) = n \mid N_1(60)] = 10$

$\Rightarrow E[N_1(60)] = 10$

$\Rightarrow n \times 0.05 = 10$

$\therefore n = 200$

c) $P_r\{N_1(60) = 5 \mid N(60) = 50\}$

[$\because$ multiplication of two poisson $dist^n$ is a Binomial $dist^n$]

$= \binom{50}{5} \cdot (0.05)^5 \cdot (0.95)^{45}$

$= 0.06584$

CamScanner

Example - 3:

Suppose that customers arrive at a counter in accordance with Poisson process with mean rate of 2 per minute ($\lambda = 2$/ minute). Then the interval between any two successive arrival's follows exponential distribution with mean $\frac{1}{\lambda} = \frac{1}{2}$ minute. What is the probability that the interval between two successive arrival is

a) More than 1 minutes.

b) 4 minute or less.

c) Between 1 and 2 minutes

d) Find the expected time untill the 9th customer.

Solution:

Suppose T represents the interarrival time between successive occurance

$$P_r\{T\} = \lambda e^{-\lambda t}$$

a) $\Pr\{T > 1\} = \int_1^\infty 2e^{-2t}\,dt$

$= 2\int_1^\infty e^{-2t}\,dt$

$= \left[-e^{-2t}\right]_1^\infty$

$\therefore \Pr\{T > 1\} = 0.1353$

b) $\Pr\{T \leq 4\} = \int_0^4 2e^{-2t}\,dt = \left[-e^{-2t}\right]_0^4$

$= 0.9996$

c) $\Pr\{1 \leq T \leq 2\} = \int_1^2 2e^{-2t}\,dt$

$= \left[-e^{-2t}\right]_1^2$

$\therefore \Pr\{1 \leq T \leq 2\} = 0.1170$

CamScanner

(d) Mean of Gamma, $\frac{n}{\lambda} = \frac{9}{2} = 4.5$ minutes

Theorem:

Let X and Y follow Poisson process with respective mean λ_1 and λ_2. Show that the conditional distribution of X given X+Y is binomial.

Proof: ~~Let~~

Since, X and Y follows Poisson process with rate λ_1 and λ_2 then $X+Y = Z$ (say) also follows a poission process with rate $\lambda_1 + \lambda_2$.

Thus,

$$Pr\{N(X) = k \mid N(Z) = n\} = \frac{Pr\{N(X) = k \text{ and } N(Z) = n\}}{Pr\{N(Z) = n\}}$$

$$= \frac{Pr\{N(X) = k \text{ and } N(Z-X) = (n-k)\}}{Pr\{N(Z) = n\}}$$

CamScanner

$$= \frac{P_r\{N(X)=k\}\ P_r\{N(Y)=n-k\}}{P_r\{N(Z)=n\}}$$

$$= \frac{\frac{e^{-\lambda_1}\lambda_1^{k}}{k!}\cdot\frac{e^{-\lambda_2}(\lambda_2)^{(n-k)}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)}\cdot(\lambda_1+\lambda_2)^{n}}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{k}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

(Proved)

Compound Poisson Process

Definition: A stochastic process $[X(t), t \geq 0]$ is said to be compound Poisson process if it can be represented as

$$X(t) = \sum_{i=1}^{N(t)} Y_i \quad ; \; t \geq 0$$

where $[N(t), t \geq 0]$ is a Poisson process and $[Y_i, i \geq 1]$ is a family of independent and identically distributed random variables which are also independent of $[N(t), t \geq 0]$. The random variable $X(t)$ is said to be a compound Poisson random variables.

Q. Find the mean and variance of compound process

Solution:

We know,

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

Mean:

$$E[X(t)] = E\{E[X(t) \mid N(t)]\} \quad \text{——— (i)}$$

Now

$$\therefore E[X(t) \mid N(t)] = E\left[\sum_{i=1}^{N(t)} Y_i \mid N(t)\right]$$

$$= E\left[\sum_{i=1}^{N(t)} Y_i\right]$$

$$= \sum_{i=1}^{N(t)} E(Y_i)$$

$$= N(t)\, E(Y_i)$$

From (i)

$$\therefore E[X(t)] = E[N(t) \cdot E(Y_i)]$$

$$= E[N(t) \cdot E(Y_i)]$$

$$\therefore E[X(t)] = \lambda t \cdot E(Y_i)$$

CS CamScanner

Variance:

$$V[X(t)] = V\left[E\{X(t) \mid N(t)\}\right] + E\left[V\{X(t) \mid N(t)\}\right] \quad \text{— (ii)}$$

$$\therefore V\{X(t) \mid N(t)\} = V\left[\sum_{i=1}^{N(t)} X_i \mid N(t)\right]$$

$$= V\left[\sum_{i=1}^{N(t)} Y_i\right]$$

$$= \sum_{i=1}^{N(t)} V(Y_i)$$

$$= N(t)\, V(Y_i)$$

$$\therefore E\left[V\{X(t) \mid N(t)\}\right] = E\left[N(t) \cdot V(Y_i)\right]$$

$$= \lambda t \; V(Y_i)$$

$$V[X(t)] = V\left[N(t)\, E(Y_i)\right] + \lambda t \cdot V(Y_i)$$

$$= \{E(Y_i)\}^2 V(N(t)) + \lambda t \cdot V(Y_i)$$

$$= \{E(Y_i)\}^2 \lambda t + \lambda t \, V(Y_i)$$

$$= \lambda t \left[\{E(Y_i)\}^2 + V(Y_i)\right]$$

$$\therefore V[X(t)] = \lambda t \, E(Y_i^2)$$

Example:

Suppose that families migrate to an area at a Poisson rate $\lambda = 2$ per week. If the number of people in each family is independent and takes on values 1, 2, 3, 4 with respective probabilites 1/6, 1/3, 1/3, 1/6 then

a) What is the expected value and variance of the number of individuals migrating to this area during a fixed five week period?

b) What is the probability that at ~~legil~~ least 240 people migrate to the area within 50 weeks.

Solution:

Y_i	1	2	3	4
$P(Y_i)$	1/6	1/3	1/3	1/6

$$E[Y_i] = \sum_{i=1}^{4} Y_i\, P(Y_i)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} + 4 \cdot \frac{1}{6}$$

$$= \frac{5}{2}$$

$$= 2.5$$

$$E[Y_i^2] = \sum_{i=1}^{4} Y_i^2 P(Y_i)$$

$$= 1 \times \frac{1}{6} + 4 \cdot \frac{1}{3} + 9 \cdot \frac{1}{3} + 16 \cdot \frac{1}{6}$$

$$= \frac{1}{6} + \frac{4}{3} + 3 + \frac{8}{3}$$

$$= 7.167$$

Let, $X(5)$ denote the number of individual migrating during a 5 week period.

$$E[X(5)] = \lambda t \, E[Y_i]$$

$$= 2 \times 5 \times 2.5$$

$$= 25$$

$$Var[X(5)] = \lambda t \, E[Y_i^2]$$

$$= 2 \times 5 \times 7.167$$

$$\therefore Var[X(5)] = 71.67$$

(b) Since, $\lambda = 2$, $E[Y_i] = 2.5$, $E[Y_i^2] = 7.167$

we see that

$$E[X(50)] = 250$$

$$E[X(50)^2] = \frac{4300}{6}$$

Now, the desired probability is,

$$P[X(50) \geq 240]$$

$$= P[X(50) \geq 239.5]$$

$$= P\left[\frac{X(50) - 250}{\sqrt{4300/6}} \geq \frac{239.5 - 250}{\sqrt{4300/6}}\right]$$

$$= P[Z \geq -0.3922]$$

$$= 1 - P(Z < -0.3922)$$

$$= 1 - \phi(-0.3922)$$

$$= 0.6525$$

Non-homogeneous Poisson Process

Derivation of Renewal Process:

Statement:

$$P_n(t) = P_r\{N(t) = n\}$$

$$= P_r\{N(t) \geq n\} - P_r\{N(t) \geq n+1\}$$

$$= P_r\{S_n \leq t\} - P_r\{S_{n+1} \leq t\}$$

$$P_n(t) = F_n(t) - F_{n+1}(t)$$

Mean function:

$$m(t) = E[N(t)]$$

$$= \sum_{n=0}^{\infty} n \cdot P_n(t)$$

$$= \sum_{n=0}^{\infty} n\left[F_n(t) - F_{n+1}(t)\right]$$

$$= \sum_{n=1}^{\infty} n\left[F_n(t) - F_{n+1}(t)\right]$$

$$= F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) + \cdots$$

$$= F_1(t) + F_2(t) + F_3(t) + \cdots$$

$$\therefore m(t) = \sum_{n=1}^{\infty} F_n(t)$$

CamScanner

Example-1:

Suppose that $P_r\{X_n = i\} = P(1-P)^{i-1}$; $i \geq 1$. That is, suppose that the interarrival distribution is geometric. Find

1) $P_r\{S_n = k\}$ 2) $P_r\{N(t) = n\}$ i.e., $P_n(t)$

Solution:

(1) $P_r\{X_n = i\} = P(1-P)^{i-1}$

$S_n = \sum X_n \sim NB(n, k)$

$$P_r\{S_n = k\} = \binom{k-1}{n-1} p^n (1-p)^{k-n} \quad ; k \geq n$$

(2) $P_n(t) = F_n(t) - F_{n+1}(t)$

$= P_r\{S_n \leq t\} - P_r\{S_{n+1} \leq t\}$

$$\therefore P_n(t) = \sum_{k=n}^{t} \binom{k-1}{n-1} p^n \cdot (1-p)^{k-n} - \sum_{k=n+1}^{t} \binom{k-1}{n-1} p^n (1-p)^{k-n}$$

CamScanner

Example 2:

Suppose we have a renewal process whose mean-value function is giving by $m(t) = 2t \; ; t > 0$

a) What is the probability that 2 or 3 ~~newew~~ renewals occur by time 10

b) What is the probability that at least 1 renewals occurs by time 10

c) What is the distribution of number of renewals by time 10

Solution:

Given,

$m(t) = 2t$

Here, $\lambda = 2$ $\qquad [\because m(t) = 2t]$

a) $$P(2 \text{ or } 3) = \frac{e^{-2t} (2t)^n}{n!}$$

$$= P[n=2] + P[n=3]$$

$$= \frac{e^{-2\times 10} (20)^2}{2!} + \frac{e^{-20} (20)^3}{3!}$$

$\therefore P(2 \text{ or } 3) = 0.00316$

CamScanner

b) $P(n \geq 1) = 1 - P(n=0)$

$$= 1 - \frac{e^{-20}(20)^0}{0!}$$

$$= 0.999$$

c)

$$P\{N(10) = n\} = \frac{e^{-20}(20)^n}{n!} \; ; n > 0$$

CamScanner

Theorem 1:

Renewal Process uniquely determines the distribution function

Proof: We know,

$$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

Taking the laplace transformation

$$m^*(s) = \sum_{n=1}^{\infty} \frac{1}{s}\left[f^*(s)\right]^n$$

$$= \frac{1}{s}\sum_{n=1}^{\infty} \left[f^*(s)\right]^n$$

$$= \frac{1}{s}\left[f^*(s) + f^{*2}(s) + f^{*3}(s) + \cdots\right]$$

$$= \frac{f^*(s)\left[1 + f^*(s) + f^{*2}(s) + \cdots\right]}{s}$$

$$m^*(s) = \frac{f^*(s)}{s\left[1 - f^*(s)\right]} \qquad \left[\because 1 + x + x^2 + \cdots = (1-x)^{-1}\right]$$

which implies that renewal process uniquely determines the distribution function.

CamScanner

Theorem 2:

Show that the renewal function also satisfies the equation

$$M(t) = F(t) + \int_0^t m(t-x)\, dF(x)$$

Proof:

$$m(t) = E[N(t)]$$

Conditioning on the occurance of first renewal

$$m(t) = \int_0^\infty E[N(t) \mid X_1 = x]\, dF(x) \; ; \qquad 0 < x, t$$

$$= \int_0^t E[N(t) \mid X_1 = x]\, dF(x)$$

$$E[N(t) \mid X_1 = x] = 1 + m(t-x)$$

$$M(t) = \int_0^t [1 + m(t-x)]\, dF(x)$$

$$= \int_0^t dF(x) + \int_0^t m(t-x)\, dF(x)$$

$$\therefore M(t) = F(t) + \int_0^t m(t-x)\, dF(x)$$

CamScanner

⊛ Second Moment and Variance of Renewal Process:

$$V[N(t)] = E[N(t)^2] - \{E[N(t)]\}^2$$

Let,

$$L(t) = E[N(t)^2]$$

$$= \sum_{n=1}^{\infty} n^2 P_n(t)$$

$$= \sum_{n=1}^{\infty} n^2 [F_n(t) - F_{n+1}(t)]$$

Taking laplace transformation,

$$L^* s = \sum_{n=1} \frac{n^2}{s} \left[\{f^*(s)\}^n - \{f^*(s)\}^{n+1} \right]$$

$$= \frac{1}{s} \left[f^*(s) - f^{*2}(s) + 4 f^{*2}(s) - 4 f^{*3}(s) + 9 f^{*3}(s) - 9 f^{*4}(s) + \cdots \right]$$

$$= \frac{f^*(s)\,[1 + f^*(s)]}{s[1 - f^*(s)]^2}$$

$$= \frac{[1 + f^*(s)]}{[1 - f^*(s)]} \cdot \frac{f^*(s)}{s[1 - f^*(s)]}$$

$$= \frac{1 + f^*(s)}{1 - f^*(s)} \cdot m^*(s)$$

CamScanner

$$L^*(s) = \frac{1 - f^*(s) + 2f^*(s)}{1 - f^*(s)} m^*(s)$$

$$= \left[1 + \frac{2f^*(s)}{1 - f^*(s)}\right] m^*(s)$$

$$= m^*(s) + \frac{2f^*(s)}{s[1 - f^*(s)]} s\, m^*(s)$$

Inverting the Laplace transformation

$$L(t) = m(t) + 2\int m(t-x)\, dm(x)$$

$\therefore$ variance, $V[N(t)] = E[N^2(t)] - \{E\{N(t)\}\}^2$

$$\therefore V[N(t)] = m(t) + 2\int m(t-x)\, dm(x) - \{m(t)\}^2$$

~~Theor~~

Theorem:

Show that the average renewal rate by time t converges with probability 1 to $\frac{1}{\mu}$ as $t \to \infty$ i.e., $\lim_{t\to\infty} \left\{ \frac{N(t)}{t} \to \frac{1}{\mu} \right\} \xrightarrow{W.P} 1$

where, $\mu = E(X_n) \leq \infty$

Proof:

$N(t) = n$

S_{N+1} is the waiting time of the 1st renewal occurance after the nth occurance

$S_N \leq t \leq S_{N+1}$

$$\Rightarrow \frac{S_N}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N+1}}{N(t)} \quad \text{——— (i)}$$

$$\therefore \frac{S_N}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} = \mu \quad \text{if } N(t) \to \infty$$

$$\frac{S_N}{N(t)} \to \mu \quad \text{if } t \to \infty$$

——— (ii)

$$\frac{S_{N+1}}{N(t)} = \frac{S_{N+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}$$

$$= \frac{S_{N+1}}{N(t)+1}\left[1 + \frac{1}{N(t)}\right]$$

$$\frac{S_{N+1}}{N(t)} \to \mu \quad \text{as} \quad t \to \infty \quad \text{——(ii)}$$

From (ii) & (iii)

$$\frac{t}{N(t)} \to \mu \quad \text{as} \quad t \to \infty$$

$$\therefore \lim_{t\to\infty} \left\{\frac{N(t)}{t} \to \frac{1}{\mu}\right\} \xrightarrow{W.P} 1 \quad , \text{ where, } \mu = E(X_n) \leq \infty$$

Example 5:

Suppose the ~~interval~~ interarrival time X_n have gamma distribution with density $f(x) = \lambda e^{-\lambda x}; x>0$

Find $P_n(t)$ and $m(t)$.

Solution:

$$P_n(t) = F_n(t) - F_{n+1}(t)$$

$$= P_r\{S_n \leq t\} - P_r\{S_{n+1} \leq t\}$$

$$= \frac{\lambda^n}{\Gamma n} e^{-\lambda x} x^{n-1} \quad ; \quad k \geq n$$

$$F_n(t) = P_n\{S_n \leq t\} = \int_0^\infty \frac{\lambda^n}{\Gamma n} e^{-\lambda x} \cdot x^{n-1} \, dF(x)$$

$$= \int_0^t \frac{\lambda^n}{\Gamma n} e^{-\lambda x} x^{n-1} \, dF(x)$$

$$= e^{-\lambda x} \sum_{r=0}^{n} \frac{(\lambda x)^r}{r!}$$

$$F_{n+1}(t) = e^{-\lambda x} \sum_{r=0}^{n+1} \frac{(\lambda x)^r}{r!}$$

$$\therefore P_n(t) = e^{-\lambda x} \sum_{r=0}^{n} \frac{(\lambda x)^r}{r!} - e^{-\lambda x} \sum_{r=0}^{n+1} \frac{(\lambda x)^r}{r!}$$

$$= e^{-\lambda x} \left[\frac{(\lambda x)^n}{n!} \right]$$

$$\therefore P_n(t) = \frac{e^{-\lambda x} (\lambda x)^n}{n!}$$

Now,

$$m^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]}$$

$$f^*(s) = L[f(x)] = E[e^{-sx}] = \int e^{-sx} \cdot \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int e^{-(s+\lambda)x} \, dx$$

CamScanner

$$f^*(s) = \frac{\lambda}{\lambda+s}$$

$$\therefore m^*(s) = \frac{\frac{\lambda}{\lambda+s}}{s\left[1-\frac{\lambda}{\lambda+s}\right]}$$

$$= \frac{\lambda}{\lambda+s} \times \frac{(\lambda+s)}{s^2}$$

$$m^*(s) = \frac{\lambda}{s^2}$$

Now, Inverting, we get

$$m(t) = \lambda t.$$

Example:

Suppose we have a renewal process with $M(t) = 3t$. Find

1) What is the probability that 2 or 3 renewals occur by time 15 ?

2) What is the probability that more than 1 renewal process for which mean value function is linear.

Solution (1):

$Pr\{N(t)=2 \text{ or } N(t)=3\}$

$= Pr\{N(t)=2\} + Pr\{N(t)=3\}$

$$= \frac{e^{-3\times15}(3\times15)^2}{2!} + \frac{e^{-3\times15}(3\times15)^3}{3!}$$

$\cong 0$

2) $Pr\{N(t)>1\} = 1 - \left[Pr\{N(t)=0\} + Pr\{N(t)=1\}\right]$

$$= 1 - \frac{e^{-45}(45)^0}{0!} - \frac{e^{-45}(45)^1}{1!}$$

$\cong 1$

Continuous Time Markov Chain

Theorem:

Suppose $\{X(t); t \geq 0\}$ be a birth and death process with

$$\lambda_n = n\lambda + \theta \quad ; \quad n \geq 0$$

and $$\mu_n = n\mu \quad ; \quad n \geq 1$$

Find the average number of member in the system at time t.

Or, Show that $M(t)$ is a queueing process.

Or, Show that $M(t) = \frac{\theta}{\lambda-\mu}\left[e^{(\lambda-\mu)t} - 1\right] + ne^{(\lambda-\mu)t}$

Proof:

$$M(t) = E[X(t)]$$

$$M(t+h) = E[X(t+h)] = E\left[E\{X(t+h) \mid X(t)\}\right] \quad \text{--- (1)}$$

Now,

$$X(t+h) = \begin{cases} X(t)+1 & \text{with probability } [X(t)\lambda + \theta]h + o(h) \\ X(t)-1 & \text{with probability } [X(t)\mu h + o(h)] \\ X(t) & \text{with probability } 1-[X(t)\lambda t + \theta + X(t)\mu]h + o(h) \end{cases}$$

$$E[X(t+h)] = \{X(t)+1\}[X(t)\lambda + \theta]h + \{X(t)-1\} X(t)\mu h +$$

$$X(t)\{1 - [X(t)\lambda + \theta + X(t)\mu]h\} + o(h)$$

$$= X(t) + X(t)\lambda h - X(t)\mu h + \theta h + o(h)$$

$$= X(t) + (\lambda-\mu)X(t)h + \theta h + o(h)$$

Taking expectations

$$M(t+h) = M(t) + (\lambda-\mu)M(t)h + \theta h + o(h)$$

$$M(t+h) - M(t) = (\lambda-\mu)M(t)h + \theta h + o(h)$$

$$\Rightarrow \lim_{h\to 0} \frac{M(t+h)-M(t)}{h} = (\lambda-\mu)M(t) + \theta + \lim_{h\to 0}\frac{o(h)}{h}$$

$$\Rightarrow M'(t) = (\lambda-\mu)M(t) + \theta \quad \text{———} \quad (2)$$

Let,

$$h(t) = (\lambda-\mu)M(t) + \theta$$

Differentiating both sides with respect to t we get,

$$h'(t) = (\lambda-\mu)M'(t)$$

$\therefore M'(t) = \frac{h'(t)}{\lambda-\mu}$

Put this value in eq^n (2) we get,

$$\frac{h'(t)}{\lambda-\mu} = \cancel{M'(t)} \; h(t)$$

$$\therefore \cancel{M'(t) = \frac{h'(t)}{\lambda-\mu}}$$

$$\Rightarrow \frac{h'(t)}{h(t)} = (\lambda-\mu)$$

Integrating both sides we get,

$$\int \frac{h'(t)}{h(t)}\,dt = (\lambda-\mu)t + c$$

$$\Rightarrow \ln[h(t)] = (\lambda-\mu)t + c$$

$$h(t) = ke^{(\lambda-\mu)t} \quad [\because k = e^c]$$

$$\Rightarrow (\lambda-\mu)M(t) + \theta = ke^{(\lambda-\mu)t} \quad \cancel{[\because k = e^c]} \qquad (3)$$

If $t = 0$, $M(0) = n$

$\therefore (\lambda-\mu)n + \theta = k$

From (3) we get

$$(\lambda-\mu)M(t) + \theta = [(\lambda-\mu)n + \theta]\,e^{(\lambda-\mu)t}$$

$$\Rightarrow (\lambda-\mu) M(t) = (\lambda-\mu) n\, e^{(\lambda-\mu)t} + \theta e^{(\lambda-\mu)t} - \theta$$

$$\Rightarrow M(t) = \frac{1}{(\lambda-\mu)}\left[(\lambda-\mu) n e^{(\lambda-\mu)t} + \theta\left\{e^{(\lambda-\mu)t} - 1\right\}\right]$$

$$M(t) = n e^{(\lambda-\mu)t} + \frac{\theta}{(\lambda-\mu)}\left[e^{(\lambda-\mu)t} - 1\right]$$

$$\therefore\ M(t) = \frac{\theta}{\lambda-\mu}\left[e^{(\lambda-\mu)t} - 1\right] + n e^{(\lambda-\mu)t}$$

(Proved)

Example:

For a general birth and death process with birth rates $\lambda_i = \lambda$ and death rates $\mu_i = \mu$, where $\mu_0 = 0$. For the process starting from state i find the expected ammount of time to enter state $i+1$

Theorem: Prove that

a) $$\lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i$$

b) $$\lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij} \quad \text{for } i \neq j$$

Proof:

a) $$1 - P_{ii}(h) = v_i(h) + o(h)$$

$$\Rightarrow \lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i + \lim_{h\to 0} \frac{o(h)}{h}$$

$$\therefore \lim_{h\to 0} \frac{1-P_{ii}(h)}{h} = v_i$$

(Proved)

b) $$P_{ij}(h) = v_i P_{ij}\, h + o(h)$$

$$\lim_{h\to 0} \frac{P_{ij}(h)}{h} = v_i P_{ij} + \lim_{h\to 0} \frac{o(h)}{h}$$

$$\therefore \lim_{h\to 0} \frac{P_{ij}(h)}{h} = q_{ij} \quad \text{for } i \neq j$$

(Proved)

CamScanner

Chapman-Kolmogorov Equation:

Statement:

Suppose $\{X(t) : t \geq 0\}$ be a continuous time Markov chain and if $s, t \geq 0$ then the Kolmogorov equation is given by

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

Proof:

$$P_{ij}(t+s) = P_r\{X(t+s) = j \mid X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P_r\{X(t+s) = j, X(t) = k \mid X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P_r\{X(t+s) = j \mid X(t) = k, X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P_r\{X(t+s) = j \mid X(t) = k\} \cdot P_r\{X(t) = k \mid X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P_{kj}(s) \cdot P_{ik}(t)$$

$$\therefore P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{kj}(s) \cdot P_{ik}(t)$$

(Proved)

CamScanner

Kolmogorov's Backward equation:

Proof: From Chapman.

$$P_{ij}(h+t) = \sum_{k=0}^{\infty} P_{ik}(h)\, P_{kj}(t)$$

$$P_{ij}(h+t) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$

$$= \sum_{k \neq i} P_{ik}(h)\, P_{kj}(t) + P_{ii}(h)\, P_{ij}(t) - P_{ij}(t)$$

$$= \sum_{k \neq i} P_{ik}(h)\, P_{kj}(t) - P_{ij}(t)\left[1 - P_{ii}(h)\right]$$

$$\Rightarrow \lim_{h \to 0} \frac{P_{ij}(h+t) - P_{ij}(t)}{h} = \sum_{k \neq i} P_{kj}(t) \lim_{h \to 0} \frac{P_{ik}(h)}{h} - P_{ij}(t) \cdot \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h}$$

$$\Rightarrow P_{ij}'(t) = \sum_{k \neq i} P_{kj}(t)\, q_{ik} - P_{ij}(t)\, v_i$$

$$\therefore P_{ij}'(t) = \sum q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

CamScanner

Kolmogorov's Forward Equation:

Proof:

From Chapman,

$$P_{ij}(h+t) = \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(h)$$

$$\Rightarrow P_{ij}(h+t) - P_{ij}(t) = \sum_{k=0}^{\infty} P_{ik}(t)\, P_{kj}(h) - P_{ij}(t)$$

$$= \sum_{k \neq j}^{\infty} P_{ik}(t)\, P_{kj}(h) + P_{ij}(t)\, P_{jj}(h) - P_{ij}(t)$$

$$= \sum_{k \neq j}^{\infty} P_{ik}(t)\, P_{kj}(h) - P_{ij}(t)\left[1 - P_{jj}(h)\right]$$

$$\Rightarrow \lim_{h\to 0} \frac{P_{ij}(h+t) - P_{ij}(t)}{h} = \sum_{k \neq j}^{\infty} P_{ik}(t) \lim_{h\to 0} \frac{P_{kj}(h)}{h} - P_{ij}(t) \lim_{h\to 0} \frac{1 - P_{jj}(h)}{h}$$

$$\Rightarrow P_{ij}'(t) = \sum_{k \neq j}^{\infty} P_{ik}(t) \lim_{h\to 0} \frac{P_{kj}(h)}{h} - P_{ij}(t) \lim_{h\to 0} \frac{1 - P_{jj}(h)}{h}$$

$$\Rightarrow P_{ij}'(t) = \sum_{k \neq j} P_{ik}(t) \cdot q_{kj} - P_{ij}(t) \cdot v_j$$

$$\left[\because \lim_{h\to 0} \frac{P_{kj}(h)}{h} = q_{kj}, \quad \because \lim_{h\to 0} \frac{1 - P_{jj}(h)}{h} = v_j\right.$$

$$\therefore P_{ij}'(t) = \sum_{k \neq i} q_{kj} \cdot P_{ik}(t) - v_j P_{ij}(t)$$

CamScanner

For Birth and Death process (Forward Kolmogorov)

$$P_{ij}'(t) = q_{j-1,j} \; P_{i,j-1}(t) + q_{j+1,j} \; P_{i,j+1}(t) - V_j P_{ij}(t) \quad \text{——} (1)$$

$$q_{j-1,j} = V_j \cdot \frac{\lambda_j}{V_j} = \lambda_j$$

$$q_{j+1,j} = V_j \cdot \frac{\mu_j}{V_j} = \mu_j$$

$$V_j = \lambda_j + \mu_j$$

From eq^n (1)

$$P_{ij}'(t) = \lambda_j \cdot P_{i,j-1} + \mu_j P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t)$$

which is the forward Kolmogorov equation for birth and death process.

* **Backward Kolmogorov Equation for birth and death process:**

$$P_{ij}'(t) = q_{i,i+1} \; P_{i+1,j}(t) + q_{i,i-1} \; P_{i-1,j}(t) - V_i P_{ij}(t) \quad \text{——} (1)$$

$$q_{i,i+1} = V_i \; P_{i,i+1} = V_i \cdot \frac{\lambda_i}{V_i} = \lambda_i$$

$$q_{i,i-1} = V_i \; P_{i,i-1} = V_i \cdot \frac{\mu_i}{V_i} = \mu_i$$

$$V_i = \lambda_i + \mu_i$$

CamScanner

From eq^n (1) we get

$$P_{ij}'(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

which is the Backward Kolmogorov equation for birth and death process.

if $i=0$ then

$$P_{0j}'(t) = \lambda_0 \cdot P_{1,j}(t) + \mu_0 P_{-1,j}(t) - (\lambda_0 + \mu_0) P_{0j}(t)$$

$$= \lambda_0 P_{1,j}(t) - \lambda_0 P_{0j}(t) \qquad [\because \mu_0 = 0]$$

$$= \lambda_0 [P_{1,j}(t) - P_{0j}(t)]$$

For pure Birth process:

In pure birth process, death rates, $\mu_i = 0$

$\therefore V_i = \mu_i + \lambda_i = \lambda_i$

Again, we know that

$$q_{i,i+1} = V_i P_{i,j+1} = V_i \cdot \frac{\lambda_i}{V_i} = \lambda_i \qquad \left[\text{Since } P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}\right]$$

We know, kolmogorv's Backward equation is

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - V_j P_{ij}(t)$$

So. for pure birth process the above equations becomes:

$$P_{ij}'(t) = \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t)$$

if $i=0$ then,

$$P_{0j}'(t) = \lambda_0 P_{1,j}(t) - \lambda_0 P_{0j}(t)$$

Theorem: For pure birth process, show that

i) $P_{ii}(t) = e^{-\lambda_i(t)}$; $t \geq 0$

ii) $P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\, ds$

Proof (i):

We know that

From Chapman forward equation

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - V_j P_{ij}(t)$$

For pure birth process

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

Let,

$j = i$

$P_{ii}'(t) = \lambda_{i-1} P_{i,i-1}(t) - \lambda_i P_{ii}(t)$ [$\because \lambda_i P_{i,i-1}(t)$ is death rate; so this part is 0]

$\Rightarrow P_{ii}'(t) = -\lambda_i P_{ii}(t)$

$\Rightarrow \dfrac{P_{ii}'(t)}{P_{ii}(t)} = -\lambda_i$

$\Rightarrow \ln[P_{ii}(t)] = -\lambda_i t + c$

$\Rightarrow P_{ii}(t) = e^{-\lambda_i t + c}$ ——— (i)

if $t = 0$,

$P_{ii}(0) = 1$

$\Rightarrow e^{c} = 1$

from (i) we get,

$P_{ii}(t) = e^{-\lambda_i t}$

(Proved)

Proof (ii):

We know

Chapman forward equation

$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - V_j P_{ij}(t)$

(P.T.O)

For pure birth process

$$P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t)$$

$$\Rightarrow P_{ij}'(t) + \lambda_j P_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t)$$

$$\Rightarrow e^{\lambda_j(t)} \left[P_{ij}'(t) + \lambda_j P_{ij}(t) \right] = e^{\lambda_j(t)} \left[\lambda_{j-1} P_{i,j-1}(t) \right]$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda_j t} \cdot P_{ij}(t) \right] = e^{\lambda_j t} \lambda_{j-1} \left[P_{i,j-1}(t) \right]$$

$$\Rightarrow \cancel{\frac{d}{dt}\left[e^{\lambda_j t} \cdot P_{ij}(t) \right]} = \cancel{e}^{\lambda_j t}$$

$$\Rightarrow e^{\cancel{\lambda_j t}} \cdot P_{ij}(t) \cancel{= \lambda_{j-1}} \int_0^t \cancel{e^{\lambda_j(s)}} \cancel{P_{i,j-1}}(s)$$

$$\Rightarrow e^{\lambda_j t} P_{ij}(t) = \lambda_{j-1} \int_0^t e^{\lambda_j t} P_{i,j-1}(t)\, dt + c$$

If $t = 0$, $P_{ij}(0) = 0$

then $c = 0$

$$\therefore e^{\lambda_j t} P_{ij}(t) = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\, ds$$

$$\therefore P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s)\, ds$$

(Proved)

CamScanner

Balance Equation:

Balance equations for solving the limiting probabilities and why it is called Balance equations?

We know that,

Kolmogorov's forward equation is,

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - V_j P_{ij}(t)$$

$$\Rightarrow \lim_{t \to \infty} P_{ij}'(t) = \sum_{k \neq j} q_{kj} \lim_{t \to \infty} P_{ik}(t) - V_j \lim_{t \to \infty} P_{ij}(t)$$

$$\Rightarrow \lim_{t \to \infty} P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_k - V_j P_j$$

$$\therefore V_j P_j = \sum_{k \neq j} q_{kj} P_k$$

and $\sum_{j} P_j = 1$

This equation is called balance equation.

$V_j P_j$ = The rate at which the process leave at state j. $\sum_{k \neq j} q_{kj} P_k$ = The rate at which the process enter at state j.

CamScanner

Limiting Probability

Balance equation

$$V_j P_j = \sum_{k \neq j} q_{kj} P_k \qquad [\because q_{kj} P_k = q_{j-1,j} P_{j-1} + q_{j+1,j} P_{j+1}$$

State	Rate of leave state j = Rate to enter state j
0	$\lambda_0 P_0 = \mu_1 P_1$
1	$(\lambda_1 + \mu_1) P_1 = \lambda_0 P_0 + \mu_2 P_2$
2	$(\lambda_2 + \mu_2) P_2 = \lambda_1 P_1 + \mu_3 P_3$
⋮	⋮
n	$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$

Adding each equations with its previous one

$\lambda_0 P_0 = \mu_1 P_1$

$\lambda_1 P_1 = \mu_2 P_2$

$\lambda_2 P_2 = \mu_3 P_3$

⋮

$\lambda_n P_n = \mu_{n+1} \cdot P_{n+1}$

Now solving each eq^n for P_0

$$P_1 = \frac{\lambda_0}{\mu_1} P_0$$

$$P_2 = \frac{\lambda_1}{\mu_2} P_1 \cdot \frac{\lambda_0}{\mu_1} \cdot P_0$$

$$P_3 = \frac{\lambda_2}{\mu_3} P_2$$

$$= \frac{\lambda_2}{\mu_3} \cdot \frac{\lambda_1 \lambda_0}{\mu_1 \cdot \mu_2} \cdot P_0 \; P_1$$

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \cdot \mu_2 \cdot \mu_3} \cdot P_0$$

$$\vdots$$

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_n} P_0$$

Now

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \quad P_0 + \sum_{n=1}^{\infty} P_n = 1$$

$$\Rightarrow \quad P_0 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_n} P_0 = 1$$

$$\Rightarrow \quad P_0 \left[1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_n} \right] = 1$$

$$\therefore \quad P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_n}}$$

Queueing Process

Derivation of M/M/1 model:

Let,

λ = arrival rate

μ = Departure rate

We know,

Balance equation,

$$V_j P_j = \sum_{k \neq j} q_{kj} P_k$$

state	$V_j P_j = \sum_{k \neq j} q_{kj} P_k$
0	$\lambda P_0 = \mu P_1$
1	$(\lambda + \mu) P_1 = \lambda P_0 + \mu P_2$
2	$(\lambda + \mu) P_2 = \lambda P_1 + \mu P_3$
⋮	⋮
n	$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$

Summing each equation with its previous one we get.

$$\lambda P_0 = \mu P_1$$

$$\lambda P_1 = \mu P_2$$

$$\lambda P_2 = \mu P_3$$

$$\vdots$$

$$\lambda P_n = \mu P_{n+1}$$

Solving each eq^n with respect to P_0

$$P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$$

$$P_2 = \frac{\lambda}{\mu} P_1 = \rho . \rho P_0 = \rho^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 = \rho^3 P_0$$

$\vdots$

$$P_n = \rho^n P_0$$

We know,

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \rho^n P_0 = 1$$

$$\Rightarrow P_0 \left[1 + \rho^2 + \rho^3 + \cdots + \rho^n \right] = 1$$

$$\Rightarrow P_0 (1-\rho)^{-1} = 1$$

$$\therefore P_0 = (1-\rho)$$

$$\therefore P_n = \rho^n (1-\rho) \quad ; \quad \rho \geq 1, \; n \geq 0$$

$\hookrightarrow$ Geometric distribution.

Property-5:

$$W_Q = W - E(S)$$

$$= W - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu}$$

$$= \frac{\mu - \mu + \lambda}{\mu(\mu-\lambda)}$$

$$\therefore W_Q = \frac{\lambda}{\mu(\mu-\lambda)}$$

~~Theorem:~~

~~The Average number of customers waiting in the queue is L_Q~~

Theorem:

The Average number of customers in the system is $L = E(N) = \frac{\rho}{1-\rho}$ and $V(N) = \frac{\rho}{(1-\rho)^2}$

Proof:

$$L = E(N) = \sum_{n=0}^{\infty} n \cdot P_n$$

$$= \sum_{n=0}^{\infty} n \cdot \rho^n (1-\rho)$$

$$= (1-\rho)\left\{ \sum_{n=0}^{\infty} n\rho^n \right.$$

CamScanner

$$= (1-\rho)[\rho + 2\rho^2 + 3\rho^3 + \cdots]$$

$$= (1-\rho)\rho[1 + 2\rho + 3\rho^2 + \cdots]$$

$$= (1-\rho)\rho(1-\rho)^{-2}$$

$$= \frac{\rho(1-\rho)}{(1-\rho)^2}$$

$$\therefore L = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$$

We know,

$$V[N] = E[N^2] - \{E(N)\}^2$$

$$E[N^2] = \sum_{n=0}^{\infty} n^2 \rho^n (1-\rho)$$

$$= (1-\rho)\sum_{n=0}^{\infty} n^2 \rho^n$$

$$= (1-\rho)\left[\cancel{\rho + 4\rho^2 + 9\rho^3 + \cdots}\right]$$

$$= (1-\rho)$$

$$= (1-\rho)\sum_{n=1}^{\infty} n(n-1)\rho^n + E(N)$$

$$= (1-\rho)(2\rho^2 + 6\rho^3 + 12\rho^4 + \cdots) + \frac{\rho}{(1-\rho)}$$

$$= (1-\rho) \times 2\rho^2 (1 + 3\rho + 6\rho^2 + \cdots) + \frac{\rho}{(1-\rho)}$$

$$= (1-\rho) \times 2\rho^2 (1-\rho)^{-3} + \frac{\rho}{1-\rho}$$

$$= \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{(1-\rho)}$$

$$= \frac{2\rho^2 + \rho(1-\rho)}{(1-\rho)^2}$$

$$= \frac{2\rho^2 + \rho - \rho^2}{(1-\rho)^2}$$

$$\therefore E[N^2] = \frac{\rho^2 + \rho}{(1-\rho)^2}$$

$$\therefore V[N] = \frac{\rho^2 + \rho}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^2}$$

$$= \frac{\rho^2 + \rho - \rho^2}{(1-\rho)^2}$$

$$\therefore V[N] = \frac{\rho}{(1-\rho)^2}$$

Theorem:

Average number of customers waiting in the queue is $L_Q = \frac{\rho^2}{(1-\rho)} = \frac{\lambda^2}{\mu(\mu-\lambda)}$

Proof:

$$L_Q = \sum_{n=1}^{\infty}(n-1)P_n$$

$$= \sum_{n=1}^{\infty}(n-1)\rho^n(1-\rho)$$

$$= (1-\rho)\left[\rho^2 + 2\rho^3 + 3\rho^4 + \cdots\right]$$

$$= (1-\rho)\rho^2\left[1 + 2\rho + 3\rho^2 + \cdots\right]$$

$$= \frac{(1-\rho)\rho^2}{(1-\rho)^2}$$

$$= \frac{\rho^2}{(1-\rho)}$$

So,

$$L_Q = \frac{\rho^2}{(1-\rho)} = \frac{(\lambda/\mu)^2}{1-\lambda/\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(Proved)

CamScanner

Theorem: The average amount of time that a customer ~~sep~~ spends for waiting in the system is $W = \frac{1}{(\mu-\lambda)} = \frac{1}{\mu(1-\rho)}$

Proof: Average amount of time a customer spends in the system is given by

$$W = E(X=x) = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x\,(\mu-\lambda)\,e^{-(\mu-\lambda)x}\,dx$$

$$= (\mu-\lambda)\int_0^\infty x^{2-1}\,e^{-(\mu-\lambda)x}\,dx$$

$$= (\mu-\lambda)\frac{\Gamma 2}{(\mu-\lambda)^2}$$

$$\Rightarrow W = \frac{1}{(\mu-\lambda)} = \frac{1}{\mu\left(1-\frac{\lambda}{\mu}\right)} = \frac{1}{\mu(1-\rho)}$$

$$\therefore W = \frac{1}{\mu(\mu-\lambda)} = \frac{1}{\mu(1-\rho)}$$

CamScanner

Relationship between L, L_Q, W, W_Q:

We know,

$$L = \frac{\rho}{1-\rho} \quad \text{——— (1)}$$

$$L_Q = \frac{\rho^2}{1-\rho} \quad \text{——— (2)}$$

$$W_Q = \frac{\rho}{\mu(1-\rho)} \quad \text{——— (3)}$$

$$W = \frac{1}{\mu(1-\rho)} \quad \text{——— (4)}$$

Now, from equation (1) we have,

$$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu(1-\rho)}$$

$$= \lambda \frac{1}{\mu(1-\rho)}$$

$$= \lambda W \qquad [\text{From } (4)]$$

$$\Rightarrow \quad L = \lambda W$$

$$\Rightarrow \quad W = \frac{L}{\lambda}$$

$$\therefore \quad W = \frac{L}{\lambda}$$

Again, we have from equation (2),

$$L_Q = \frac{\rho^2}{1-\rho} = \frac{\lambda}{\mu} \cdot \frac{\rho}{1-\rho}$$

$$= \lambda \cdot \frac{\rho}{\mu(1-\rho)}$$

$$= \lambda W_Q \qquad \text{[From (3)]}$$

$$\Rightarrow L_Q = \lambda W_Q$$

$$\Rightarrow W_Q = \frac{L_Q}{\lambda}$$

$$\therefore W_Q = \frac{L_Q}{\lambda}$$

Relationship between μ and λ

We know that

$$P_0 \geq 0$$

$$\Rightarrow 1-\rho \geq 0$$

$$\Rightarrow 1-\frac{\lambda}{\mu} \geq 0$$

$$\Rightarrow \frac{\lambda}{\mu} \leq 1 \qquad \Rightarrow \rho \leq 1$$

$$\therefore \lambda \leq \mu$$

$\cancel{\rho \leq 1}$

That is the service rate must be greater than arrival rate.

CamScanner

Example

Solution

Suppose λ represents that rate of arrival and μ represents the rate of service time.

Here, the customers arrive at a Poisson rate of $\lambda = \frac{1}{12}$ per minute. The service time is exponential at a rate of $\mu = \frac{1}{8}$ per minute.

So, we have, $\rho = \frac{\lambda}{\mu} = \frac{8}{12} = \frac{2}{3}$

the queueing model M/M/1 that there are n customers in the supermarket is given by

$$P_n = \rho^n (1-\rho) \quad ; \quad n \geq 0$$

(1) there is no customer in the supermarket is given by

$$P_0 = 1 - \rho = 1 - \frac{2}{3} = \frac{1}{3} = 0.333$$

$\therefore$ So, about 33% times the supermarket will be free from customers.

(2) Proportion of time the supermarket is busy is given by

$$1 - P_0 = 1 - 1 - (1-\rho) = \rho$$

$$= \frac{2}{3}$$

$\therefore$ So, about 66% time the supermaket will be busy

(3) Average number of customers is given by

$$L = \frac{\rho}{1-\rho} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$$

$\therefore$ So, on average there will be 2 customers per minute in the supermarket.

(4) Average amount of time spend in the queue is given by

$$W_Q = \frac{L_Q}{\lambda} = \frac{\frac{\rho^2}{1-\rho}}{\lambda}$$

$$= \frac{\frac{(2/3)^2}{1-\frac{2}{3}}}{\frac{1}{12}} = 16$$

CamScanner

So, average amount of time a customer spends in the queue to get into the super market is 16.

Example: Suppose that in the

Example:

Solve:

The arrival rate increases as $= \frac{1}{12} \times 20\% = \frac{1}{60}$ per minute.

Then the present arrival rate is $\lambda = \frac{1}{12} + \frac{1}{60} = \frac{1}{10}$ per minute.

So, we have $\rho = \frac{\lambda}{\mu} = \frac{8}{10} = \frac{4}{5}$

(1) So, we have $L = \frac{\rho}{1-\rho} = \frac{\frac{4}{5}}{1-\frac{4}{5}} = 4$

Hence, an increase of 20% in the arrival rate doubled the average number of customers in the system is 4.

(2) Again we have,

$W = \frac{L}{\lambda} = \frac{4}{\frac{1}{10}} = 40$

CamScanner

Hence, in an increase of the 20% in the arrival rate, the average amount of a time a customer has to spend in the system is 40.

Example:

Solution: Here, the customers arrive at a Poisson rate of $\lambda = 8$ per hour. The service time is exponential at a rate of $\mu = 10$ per hour.

So, we have, $\rho = \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$

(1) An arrival customer has to wait on arrival if the system is not empty

$$P(N \geq 1) = 1 - P(N=0)$$
$$= 1 - P_0$$
$$= \rho$$
$$= 0.8$$

Hence, the probability that an arriving customer has to wait is given by 0.8.

CamScanner

(2) Again, we have

$$P_4 = (1-\rho)\,\rho^4$$

$$= (1-0.8)\times(0.8)^4$$

$$= 0.08192$$

Hence, the probability that 4 customers are in the system is given by 0.08192

(3) Again, 15 min $= \frac{1}{4}$ hours $= 0.25$ hour

$$P(X<0.25) = \int_0^{0.25} (\mu-\lambda)\, e^{-(\mu-\lambda)x}\, dx$$

$$= -\left[e^{-(\mu-\lambda)x} \right]_0^{0.25}$$

$$= 1 - e^{-(\mu-\lambda)0.25}$$

$$= 1 - e^{-(10-8)0.25}$$

$$= 0.3935$$

Hence, the probability that an arriving customer has to spend less than 15 minutes in the bank is given by 0.3935

CamScanner

(4) Again, we ~~to~~ get, $1 - P_0 = \rho = 0.8$

So, the function of time the counter will busy is 0.8

F-2022

4/c From the given data, we have

$\lambda = 15$ students/ hour

mean-servic time = 3 minutes

$$= \frac{3}{60} = 0.05 \text{ hour}$$

$$\therefore \mu = \frac{1}{0.05} = 20 \text{ students / hour}$$

Traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{15}{20} = 0.75$$

(i) ~~W_q~~ ~~=~~ Average time a student spends in the queue (W_q)

$$\cancel{W_q} = \cancel{\frac{\lambda}{\mu(\mu-\lambda)}} \quad \frac{\rho}{\mu(1-\rho)} \quad \cancel{= \frac{0.75}{20(1-0.75)}}$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{15}{20(20-15)} = 0.15 \text{ hour}$$

$\therefore W_q = 0.15 \times 60 = 9$ minutes.

$\therefore$ the average time in the queue is 9 minutes.

(ii) Average time a student waits before getting service

$$W = \frac{1}{\mu(1-\rho)} = \frac{1}{20(1-0.75)}$$

$$= 0.2 \text{ hour}$$

$$\therefore W = 12 \text{ minutes}$$

So, average time a student waits before getting service is 12 minutes.

iii) Serivice utilization rate

$$\rho = \frac{\lambda}{\mu} = 0.75 = 75\%$$

(iv) $P(X > t) = e^{-(\mu-\lambda)t}$

Here, $t = 3$ min $= \frac{3}{60} = 0.05$ hours

CamScanner

$$\therefore P(X>0.05) = \int_0^\infty e^{-(20-15)\times 0.05}$$

$$= e^{-0.25}$$

$$\therefore P(X>0.05) = 0.7788$$

$\therefore$ The probability that an arriving student has to wait more than 3 minutes in the bank is given by 0.7788.

Final - 2021

4/c

(a) Given, arrival rate $\lambda = 7$ customers per minute

Average service time $= \frac{1}{10}$

$\therefore$ service rate.

$$\mu = \frac{1}{1/10} = 10 \text{ customers per minute}$$

(i) Server utilization rate $\rho = \frac{\lambda}{\mu} = \frac{7}{10} = 0.7$

Comment: Since $\rho < 1$, the system is stable. the server is busy 70% of the time and idle 30% of the time. Hence congestion is moderate and the system can handle arrivals efficiently.

(ii) Average number of customers visiting the store is

$$\cancel{L = \frac{\lambda}{\mu-\lambda} = \frac{7}{10-7} = 2.33}$$

$$L = \frac{\rho}{1-\rho} = \frac{0.7}{1-0.7} = 2.33$$

$\therefore L \approx 2.33$ customers.

(iii) $P(X > t) = e^{-(\mu-\lambda)t}$

~~Given, t = 3~~,

$$\therefore P(X > 3) = e^{-(10-7)\times 3}$$

$$= e^{-9}$$

$$= 0.000123$$

$\therefore$ The prob^y that a customer has to wait more than 3 minutes to get the desired service is 0.000123.

CamScanner